

The universal phase space of AdS_3 gravity

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Abstract

We describe what can be called the “universal” phase space of AdS_3 gravity, in which the moduli spaces of globally hyperbolic AdS spacetimes with compact spatial sections, as well as the moduli spaces of multi black hole spacetimes are realized as submanifolds. The universal phase space is parameterized by two copies of the universal Teichmüller space $\mathcal{T}(1)$. The parameterization is obtained from the correspondence between maximal surfaces in AdS_3 and quasi-symmetric homeomorphisms of the unit circle. This yields a symplectic map $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$ generalizing the well-known Mess map in the compact spatial surface setting. We also relate our parametrization to the Chern-Simons formulation of 2+1 gravity and, infinitesimally, to the holographic (Fefferman-Graham) description. In particular, we relate the charges arising in the holographic description (such as the mass and angular momentum of an AdS_3 spacetime) to the periods of the quadratic differentials arising via the Bers embedding of $\mathcal{T}(1) \times \mathcal{T}(1)$.

1 Introduction

Since the discoveries by Brown and Henneaux [1] that the group of symmetries of an asymptotically AdS_3 spacetime is a centrally extended conformal group in two dimensions, and then by Banados, Teitelboim and Zanelli [2] that black holes can exist in such spacetimes, the subject of negative cosmological constant gravity in 2+1 dimensions continues to fascinate researchers. The result [1] is now considered to be a precursor of the AdS/CFT correspondence of string theory [3], and the value of the central charge determined in [1] is an essential ingredient of the conformal field theoretic explanation [4] of the microscopic origin of the black hole entropy.

The $\text{AdS}_3/\text{CFT}_2$ story is reasonably well-understood in the string theory setting of 3-dimensional gravity coupled to a large number of fields of string (and extra dimensional) origin. At the same time, the question of whether there is a CFT dual to pure AdS_3 gravity

remains open, see [5] and [6] for the most recent (unsuccessful) attempts in this direction. In particular, the attempt [6] to construct the genus one would-be CFT partition function by summing over the modular images of the partition function of pure AdS leads to discouraging conclusions. It thus appears that pure AdS₃ gravity either does not have enough “states” to account for the BH entropy microscopically, or that the known such states cannot be consistently put together into some CFT structure.

The current lack of understanding of pure AdS₃ gravity quantum mechanically is particularly surprising given the fact that, in a sense, the theory is trivial since pure gravity in 2+1 dimensions does not have any propagating degrees of freedom. In the setting of compact spatial sections the phase space of 2+1 gravity (i.e. the space of constant curvature metrics in a $\mathbb{R} \times \Sigma$, with Σ a genus $g > 1$ Riemann surface) is easy to describe (for all values of the cosmological constant). The constant mean curvature foliation of such a spacetime is particularly useful for this purpose. One finds, see [7] and also [8] for a more recent description, that the phase space is the cotangent bundle over the Teichmüller space of the spatial slice (for any value of the cosmological constant). The zero cosmological constant result [7] also follows quite straightforwardly from the Chern-Simons (CS) description given in [9]. In the setting of AdS₃ manifolds with compact spatial slices, there is yet another description of the same phase space, first discovered by Mess [10]. This is given by two copies of the Teichmüller space of the spatial slice, or, equivalently, by two hyperbolic metrics on the spatial slice Riemann surface. The Mess description is related to the Chern-Simons description of AdS₃ gravity in terms of two copies of $SL(2, \mathbb{R})$ CS theory.

It appears sensible to tackle the problem of quantum gravity as a problem of quantization of the arising classical phase space. One could argue that this approach is unlikely to succeed in 3+1 and higher dimensions, where the phase spaces that arise this way are infinite dimensional (because of the existence of local excitations — gravitational waves). However, in the setting of 2+1 gravity, at least in the setting of spacetimes with compact spatial slices one deals with a finite-dimensional dynamical system and the problem of quantum gravity seems to reduce to a problem from quantum mechanics. In spite of this being a tractable problem, the immediate worry with this approach is that the Hilbert space of quantum states one can obtain by quantizing such a finite-dimensional phase space would not be sufficiently large to account for the black hole entropy.

At the same time, in the context of black holes one should consider non-compact spatial slices. Let us discuss what kind of the classical phase space should be expected to arise in this context. The non-compact setting is somewhat less understood. On one hand, we now know that there is not just the simple BTZ BH [2], but also a much more involved zoo of multi-black hole (MBH) spacetimes first described in [11]. A rather general description of such MBH’s using causal diamonds at their conformal infinity is given in [12]. As a byproduct of a construction in [13] using earthquakes, another description of MBH geometries is also available. These descriptions show that, like in the compact spatial slice setting with its Mess parameterization, the geometry of multi-black holes continues to be parameterized by two hyperbolic metrics on their spatial slice (or, equivalently, by the cotangent bundle of the corresponding Teichmüller space). The main difference with the compact setting is that the spatial slices are now Riemann surfaces with a geodesic boundary (or with hyperbolic ends attached), and there are now additional moduli, namely the sizes of the boundary components. These new length parameters, two for each boundary component (because there are two hyperbolic metrics involved in the parameterization) determine the geometrical charac-

teristics of the corresponding black hole horizon, such as its length and angular velocity. An explicit formula of this sort can be found in e.g. [13], see formula (1) of the first (arxiv) version of this paper. All in all, there is a reasonable understanding of the geometry of the multi-black hole spacetimes, as well as an efficient parameterization of these spacetime by two copies of the Teichmüller space of Riemann surfaces with boundaries. It thus might seem that the phase space of non-compact spatial slices AdS_3 gravity is as finite dimensional as in the compact setting.

It is however clear that a geometrical description of the multi-black hole spacetime is just half of the story. Indeed, the non-triviality of 2+1 gravity in asymptotically AdS setting comes from the fact [1] that the diffeomorphisms that are asymptotically non-trivial should no longer be interpreted as gauge. Indeed, they map one asymptotically AdS spacetime into a non-equivalent one. Thus, asymptotic symmetries applied e.g. to the AdS_3 create an infinitely large class of asymptotically AdS_3 spacetimes described by Brown and Henneaux [1]. The phase space consisting of all such spacetimes is then infinite dimensional and the problem of its quantization therefore becomes much more non-trivial than in the compact spatial slice setting. It could be that the CFT dual of pure 2+1 gravity can be discovered by quantizing this phase space. And indeed, the states obtained from the AdS “vacuum” by an action of the Virasoro generators is what was summed over in [6] in the authors’ attempt to build the genus one pure gravity partition function.

We can now formulate the main objective of this paper. Our main aim is to give a description of the phase space of AdS_3 gravity that is equally applicable to both compact and non-compact spatial section spacetimes. At the same time, we would like our description to include the Brown-Henneaux asymptotic “excitations”. As we shall see, there is a “universal” way of doing so, where one constructs what can be called the universal phase space, in which all the moduli spaces of fixed spatial topology are realized as submanifolds. We achieve this in the same way as in the context of the universal Teichmüller space, where the fixed topology Teichmüller spaces are realized as (complex) submanifolds of the universal Teichmüller space. The construction of this paper can then be seen as a generalization of the description of [10] to the setting of the universal Teichmüller space.

We want to emphasize that we do not consider here the quantum theory that would arise by quantizing the classical phase space of 2+1 gravity. This is left to future studies. Rather, our main aim here is to describe the phase space in as explicit terms as possible, thus setting the stage for its quantization. We shall see that the universal phase space is extremely non-trivial, and is parameterized by two copies $\mathcal{T}(1) \times \mathcal{T}(1)$ of the universal Teichmüller space $\mathcal{T}(1)$, or, equivalently, the cotangent bundle $T^*\mathcal{T}(1)$ over $\mathcal{T}(1)$. This generalizes the Mess’s description [10] of the compact spatial slice setting, where there is similarly two equivalent parameterizations of the moduli space of spacetimes.

Let us briefly indicate how the universal Teichmüller space comes about. In one possible definition of the latter, this is the space of quasisymmetric homeomorphisms of the unit circle (such homeomorphisms are boundary values of quasi-conformal maps from the unit disc to itself). Thus, in very general terms, the universal phase space of AdS_3 gravity is parameterized by two functions on the circle. To obtain the moduli space of spacetimes of fixed spatial topology, e.g. that of fixed topological type multi-black holes, one imposes the condition that the functions in question are invariant under a suitable discrete subgroup of the group of Möbius transformations. In the case of multi-black hole spacetimes this produces a moduli space that is still infinite-dimensional and that includes the Brown-Henneaux “excitations”

in all asymptotic regions. The cardinality is that of a pair of functions for each asymptotic region. While the freedom of prescribing two functions on the circle could be anticipated already from the Fefferman-Graham type description, see below, one novelty of our construction is that the phase space includes all possible multi-black hole spacetimes. Another novelty of the constructions of this paper is a precise characterization of which functions on the circle are relevant in the context of AdS_3 gravity. Indeed, our description in terms of two points in the universal Teichmüller space shows that these are functions coming from quasisymmetric maps of the unit circle. This is a larger class than that of smooth maps.

As we have already mentioned, the fact that in the non-compact spatial slice setting the phase space becomes an infinite dimensional space of certain (pairs of) homeomorphisms of \mathbb{S}^1 can be expected already from the AdS/CFT perspective. In fact, we know that a possible description of an asymptotically AdS spacetime is in terms of an expansion of the spacetime metric in a neighbourhood of the conformal boundary, see e.g. [14, 15] for such expansions in the AdS_3 context. For any asymptotically AdS_3 spacetime one can find the so called Fefferman-Graham coordinates in a neighbourhood of (a component of) the conformal boundary where the bulk metric takes the form

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}(g_{(0)} + \rho^2 g_{(2)} + \rho^4 g_{(4)})$$

Here $g_{(0)}$ is a representative of the conformal class on the conformal boundary and

$$g_{(2)} = \frac{1}{2}(R_{(0)}g_{(0)} + T), \quad g_{(4)} = \frac{1}{4}g_{(2)}g_{(0)}^{-1}g_{(2)}$$

with $R_{(0)}$ the Ricci scalar of $g_{(0)}$ and T the quasilocal stress-tensor [16, 17]. Note that for fixed $g_{(0)}$ the only freedom in specifying the space time metric are the components of T .

For a flat boundary metric (which is always achievable by choosing ρ appropriately), the most general quasilocal stress tensor can be written

$$T = adt^2 + 2bdtd\theta + ad\theta^2, \tag{1}$$

with a and b given by sum and difference of two arbitrary chiral functions

$$a(t, \theta) = a_+(t + \theta) + a_-(t - \theta), \quad b(t, \theta) = a_+(t + \theta) - a_-(t - \theta), \tag{2}$$

and the spacetime metric becomes

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{4\rho^2}(-dt^2 + d\theta^2) + \frac{1}{2}(adt^2 + 2bdtd\theta + ad\theta^2) + \frac{\rho^2}{4}(a^2 - b^2)(-dt^2 + d\theta^2). \tag{3}$$

Note that the possibility to write down the Fefferman-Graham type expansion in a closed form (with a finite number of terms) is peculiar to 2+1 dimensions, and is due to the absence of any local degrees of freedom in this theory. It thus becomes clear that asymptotically AdS_3 spacetimes are parametrized by certain pairs of functions of $t \pm \theta$. The above description is, however, not entirely satisfactory. Indeed, the Fefferman-Graham coordinate ρ extends only over a portion of the spacetime near its conformal boundary. Thus, only very little control over what happens inside the spacetime is available. In particular, it is not possible to know whether a spacetime (3) contains any non-trivial topology. It is also very hard to

characterize those choices of a_{\pm} that lead to non-singular spacetimes. For all these reasons the description (3), while indicating that there is some infinite-dimensionality to be expected, is not a satisfactory description of the phase space of asymptotically AdS_3 gravity.

As we will show in this paper, a description of the phase space that overcomes these difficulties is possible by using embedded maximal surfaces. Indeed, one particularly powerful description of the compact spatial slice situation is based on maximal surfaces, see [8]. It can be shown that each AdS_3 spacetime with a compact spatial slice (such spacetime were referred to as globally hyperbolic maximally compact (GHMC) AdS in [8]) contains a unique maximal surface. The first and second fundamental forms induced on such a surface then become the configurational and momentum variable. It can be shown that the free data are those of a conformal structure and a certain quadratic differential on the maximal surface, and these together parameterize a point in the cotangent bundle of the Teichmüller space of the Cauchy surface. The data on the maximal surface can in turn be used to produce two hyperbolic metrics via a generalized Gauss map, and thus two points in the Teichmüller space, and this way one obtains an explicit realization of the Mess parameterization [10].

The present work more or less generalizes the above compact case description to the non-compact setting. Thus, similar to the construction described in [8], we shall present two parameterizations of the phase space. One of them works with metrics and quadratic differentials on the disc, and thus provides an analog of the cotangent bundle description. The other works with two metrics on the disc, and is the analog of the two copies of the Teichmüller space description. The relation between both parametrizations is obtained explicitly from the harmonic decomposition of quasiconformal minimal Lagrangian diffeomorphisms of the disc, thus generalizing the Mess map to a symplectomorphism $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$. Our main mathematical result is a description of this highly non-trivial map, and a proof of the fact that it is one-to-one. We also give a “physicists” proof that this map is a symplectomorphism, by utilizing the Chern-Simons description of 2+1 gravity.

The construction of the map $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$ in this paper builds on and extends those in [18] and [19]. Thus, there is not much new mathematics in this work. Rather, we take some results obtained by mathematicians (with different aims), and use them to describe the phase space of AdS_3 gravity. In particular, our description is based on the result in [18] that proved the existence and uniqueness of maximal surfaces in AdS_3 with a given boundary curve. This boundary curve is, in turn, parameterized by a single quasisymmetric homeomorphism on the circle, so there is a one-to-one correspondence between quasisymmetric homeomorphisms of the circle (i.e. points in the universal Teichmüller space) and maximal surfaces in AdS_3 . We use these, as well as some results from the universal Teichmüller literature, to describe the phase space, seen as the space of all AdS_3 spacetimes, in terms of deformations of the domain of dependence of a totally geodesic spacelike surface in AdS_3 . The part of our work related to the splitting of minimal Lagrangian diffeomorphisms into harmonic maps is based on [19].

One non-obvious point of our construction, which is also where we depart from the works cited above, is the existence of two independent directions in our phase space. Thus, the work [18] makes it clear that a single point in $\mathcal{T}(1)$ gives rise to a maximal surface in AdS_3 , which then comes with its first and second fundamental forms (induced by the embedding in AdS_3). Thus, it can appear that a single quasisymmetric homeomorphism on the circle (single point in $\mathcal{T}(1)$) is sufficient to specify all of the “initial” data necessary for the maximal surface description. The question is then where does the second direction in our phase space,

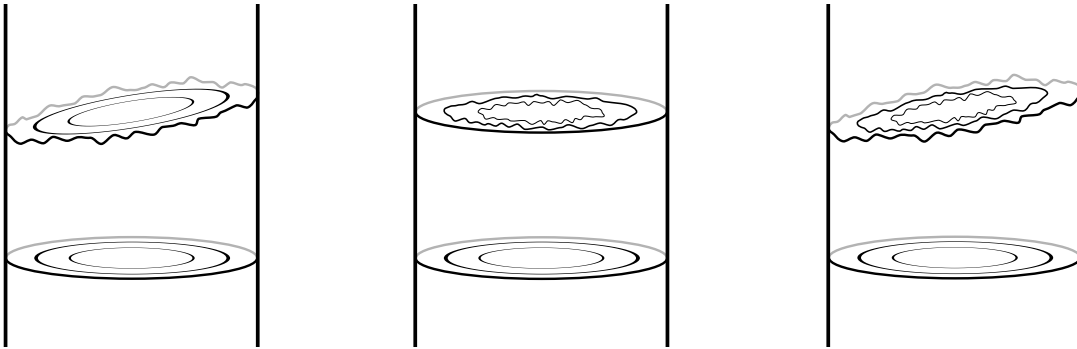


Figure 1: Two deformation directions. One (left figure) corresponds to deforming the curve along which the maximal surface intersects the boundary. The other (middle figure) corresponds to deforming the complex structure on the maximal surface, or, geometrically, to deforming the constant “radial” coordinate foliations of the surface. A general point in the phase space deforms both the curve at infinity as well as the constant radial coordinate foliation of the maximal surface (right figure).

i.e. a second point in $\mathcal{T}(1)$ comes from. As we shall describe in more details below, this other direction comes from the possibility of an additional quasiconformal diffeomorphism on the maximal surface. Being a diffeomorphism it does not change the first and second fundamental forms on the maximal surface. But being asymptotically non-trivial diffeomorphism it gives rise to deformations that has to be considered as non-gauge. And we shall verify that the two types of deformations — the geometric ones corresponding to changing the curve along which the maximal surface intersects the boundary, and the non-geometric one corresponding to just performing an asymptotically non-trivial diffeomorphism on the maximal surface — are canonically conjugate to each other in the symplectic structure induced by the gravity action. Thus, both are equally important as far as AdS_3 gravity is concerned. The two phase space directions — those deforming the curve along which the maximal surface intersects the cylinder at infinity and those deforming the complex coordinate on the maximal surface — are graphically depicted in Figure 1.

Another remark is that, as in the compact setting, the parameterization by two copies of the (universal) Teichmüller space is related to the Chern-Simons formulation of 2+1 gravity introduced in [9]. Indeed, we recall that, for AdS spacetimes, the Einstein-Hilbert action can be written as two copies of $\text{SL}(2, \mathbb{R})$ Chern-Simons action. Every AdS metrics can, therefore, be described by an associated pair of flat $\text{SL}(2, \mathbb{R})$ connections. Then a simple explicit computation shows that, in our parametrization, each copy of $\mathcal{T}(1)$ corresponds to one of these connections. In fact, this relation to the CS formulation is the easiest way to understand why the generalized Mess map $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$ is a symplectomorphism.

The outline of the present paper is as follows. In section 2 we give a brief review of the compact case. Section 3 deals with the maximal surfaces in AdS_3 . The construction of the phase space of globally hyperbolic AdS spacetimes is described in section 4. We present the generalized Mess map in section 5. A relation to the holographic description is worked out in sections 6, 7, 8. We finish with a discussion. For those not familiar with (universal) Teichmüller theory we present a quick overview in the appendix.

2 Compact spatial topology

In this section we consider globally hyperbolic AdS_3 spacetimes $M = \mathbb{R} \times \Sigma$ whose Cauchy surface is a genus $g \geq 2$ Riemann surface Σ . In the Hamiltonian formulation of general relativity one foliates spacetime by spacelike hypersurfaces. The spacetime metric is then described in terms of the first and second fundamental forms (I, II) of the initial Cauchy surface Σ . The first and second fundamental forms have to satisfy certain relations, known as the Gauss-Codazzi equations (or as the Hamiltonian and momentum constraints in the GR community). In [7], the phase space of flat 2+1 gravity was shown to be parametrized by the cotangent bundle over Teichmüller space of the initial surface by choosing a foliation by constant mean curvature surfaces. In isothermal coordinates on Σ , associated with its conformal structure, we can write

$$I = e^{2\varphi}|dz^2|, \quad II = \frac{1}{2}(qdz^2 + \bar{q}d\bar{z}^2) + \frac{H}{2}e^{2\varphi}|dz^2|,$$

where H is the mean curvature of Σ . Codazzi equation imposes holomorphicity of the quadratic differential qdz^2 defined by the traceless part of II and Gauss equation becomes an equation for the “Liouville” field φ . This equation becomes particularly simple on a maximal surface $H = 0$ and reads:

$$4\partial_z\partial_{\bar{z}}\varphi = e^{2\varphi} - e^{-2\varphi}|q|^2. \quad (4)$$

Note that this is the equation relevant for the AdS_3 setting. Similar equations (with some sign changes) hold also in the positive or zero scalar curvature settings. However, what is very special about the AdS_3 situation is that the existence and uniqueness of the solution of the above Gauss equation holds. This is related to the existence and uniqueness of a maximal surface in any globally hyperbolic AdS_3 spacetime (here with compact spatial slices). Given the existence and uniqueness of φ satisfying (4), the pair (I, II) is completely determined by a conformal structure z and a holomorphic quadratic differential qdz^2 , a point in $T^*\mathcal{T}(\Sigma)$. This gives an efficient explicit description of the spacetime geometry as parameterized by the data on the maximal surface. Indeed, the 3-metric can be written in the equidistant coordinates to the maximal surface as

$$ds^2 = -d\tau^2 + \cos^2 \tau I + 2 \sin \tau \cos \tau II + \sin^2 \tau II I^{-1} II. \quad (5)$$

A direct computation then shows the gravitational symplectic structure agrees with the canonical cotangent bundle one, see [8] for more details. We thus have a description of the phase space of AdS_3 gravity in the compact spatial slice setting as $T^*\mathcal{T}(\Sigma)$.

In [10] Mess obtained another parametrization of the same spacetimes, by two copies of Teichmüller space $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$. His construction can be understood as follows. In the projective model, AdS_3 can be seen as the image of the quadric $X = \{x \in \mathbb{R}^{2,2}; \langle x, x \rangle = -1\}$, with its induced metric, under the projection $\pi : X \rightarrow \mathbb{RP}^3$,

$$\text{AdS}_3 = \pi(X) = \{[x] \in \mathbb{RP}^3; \langle x, x \rangle < 0\}.$$

A point on AdS_3 is then in correspondence with a line in $\mathbb{R}^{2,2}$ passing through a point in the quadric X and the origin. The boundary of AdS_3 , the projective quadric

$$\partial\text{AdS}_3 = \{[x] \in \mathbb{RP}^3; \langle x, x \rangle = 0\},$$

is known to be foliated by two families of projective lines \mathcal{L}_+ and \mathcal{L}_- (corresponding to the left and right null geodesics). Since each line of one family intersects a line of the other family a single time, this provides an identification of ∂AdS_3 with the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

Now, a line in \mathcal{L}_+ (resp. \mathcal{L}_-) meets the boundary of any spacelike surface at a single point so one can use these “left” and “right” families to define maps between the boundaries of any pair of spacelike surfaces. Let us now require these spacelike surfaces to be totally geodesic. Given a pair of such totally geodesic spacelike surfaces P_0 and P , let $\pi_+, \pi_- : \partial P \rightarrow \partial P_0$ be the “left” and “right” maps of their boundaries. These then uniquely extend to isometries $\Phi_+^P, \Phi_-^P : P \rightarrow P_0$ of AdS_3 sending P to P_0 . Now taking an arbitrary spacelike surface S , one can associate to S a pair of diffeomorphisms $\Phi_+, \Phi_- : S \rightarrow P_0$ by

$$\Phi_+(x) = \Phi_+^{P(x)}(x), \quad \Phi_-(x) = \Phi_-^{P(x)}(x),$$

where $P(x)$ is the totally geodesic spacelike surface tangent to S at x . If taken modulo an overall isometry in AdS_3 , this construction is independent of the choice of P_0 .

Given an GHMC AdS spacetime M , let Σ be some smooth embedded spacelike surface and consider its lift S into the universal cover of M , which is AdS_3 . Taking the pull-back of the hyperbolic metric on P_0 by Φ_+ and Φ_- defines two hyperbolic metrics on S , which in turn descend to hyperbolic metrics I_+ and I_- on Σ , thus defining a point in $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$. This construction can be applied to any spacelike surface Σ in M and it can be shown that the point in $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ one gets is independent of this choice. When Σ is maximal (or at least of constant mean curvature), the data on Σ also gives rise to a point in $T^*\mathcal{T}(\Sigma)$, as we reviewed above. This defines what can be referred to as the Mess map

$$\text{Mess} : T^*\mathcal{T}(\Sigma) \rightarrow \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \quad (6)$$

that takes (I, II) on the maximal surface into (I_+, I_-) . It can be shown that this map is a bijection, see [8]. Thus, there are two equivalent descriptions of the moduli space of globally hyperbolic AdS_3 spacetimes with spatial sections of fixed topology. One is given by $T^*\mathcal{T}(\Sigma)$, the other by $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$.

By a calculation, also available in [8], the Mess map can be explicitly described as follows

$$I_{\pm} = I(E \pm JI^{-1}II \cdot, E \pm JI^{-1}II \cdot), \quad (7)$$

where E is the identity operator on $T\Sigma$ and J is the complex structure associated with the conformal structure of Σ . These two metrics on Σ are hyperbolic provided I, II satisfy the Gauss and Codazzi equations. We therefore get a point in $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$. The relation to the previous description is that each of the two metrics in (7) is the pull-back of the hyperbolic metric on the geodesic disc P_0 via the maps Φ_{\pm} . We note that each of the maps Φ_{\pm} is harmonic, with their Hopf differentials adding to zero. The maps Φ_{\pm} can be referred to as the generalized Gauss map, see e.g. [19]. This terminology is legitimate given the resemblance of the construction of the metrics I_{\pm} with the famous Gauss map between the data on a constant mean curvature surface in $\mathbb{R}^{1,2}$ and hyperbolic metrics.

The work [8] also described an explicit inverse of the Mess map, by providing a map between a pair I_{\pm} of hyperbolic metrics on Σ and the first and second fundamental forms of the *maximal* surface in the spacetime M that corresponds to I_{\pm} . This map uses the existence and uniqueness of a minimal Lagrangian diffeomorphism between a surface Σ

equipped with the “left” and “right” hyperbolic metrics I_{\pm} . We shall denote this map by $F : (\Sigma, I_+) \rightarrow (\Sigma, I_-)$. It is an area preserving diffeomorphism (hence the term Lagrangian) whose graph is minimal in the product $(\Sigma \times \Sigma, I_+ \times I_-)$ (hence the term minimal). As is reviewed in [8], the existence of a minimal Lagrangian F is equivalent to the existence of an operator $b : T\Sigma \rightarrow T\Sigma$ satisfying

1. $\det b = 1$;
2. b is self-adjoint with respect to I_+ ;
3. $d^{\nabla^+} b = 0$, where ∇^+ is the Levi-Civita connection of I_+ ;
4. $F^* I_- = I_+(b \cdot, b \cdot)$.

In terms of b one can construct a metric and a symmetric bilinear form on Σ

$$I = \frac{1}{4} I_+(E + b \cdot, E + b \cdot), \quad II = -IJ(E + b)^{-1}(E - b) \quad (8)$$

which satisfy the Gauss-Codazzi equations. Thus, the problem of constructing the inverse map reduces to the problem of determining the map b . Once b is known, the first and second fundamental form I, II obtained by the above formulas are those of the *maximal* surface in the spacetime corresponding to the pair I_{\pm} . An explicit expression for the spacetime metric is then given by (5) providing an efficient parameterization of the space of globally hyperbolic AdS_3 spacetime (with compact spatial slices) by two copies of the Teichmüller space. This description of the inverse of the Mess map admits a direct generalization to the non-compact setting of interest to us, and will play an important role in the next sections.

We also note that the gravitational symplectic structure, evaluated in the parameterization $T^*\mathcal{T}(\Sigma)$, is just the canonical cotangent bundle symplectic structure, see [8] for a simple calculation that demonstrates this. It can also be verified that the map Mess is a symplectomorphism. The easiest way to see this is to use the Chern-Simons description. In this description the left and right metrics of the Mess parameterization encode the monodromies of the left and right $\text{SL}(2, \mathbb{R})$ connections on Σ . Since CS formulation provides an equivalent description of AdS_3 gravity, and the symplectic structure of $\text{SL}(2, \mathbb{R})$ CS theory reduces to the Weil-Petersson symplectic structure on $\mathcal{T}(\Sigma)$, the Mess map must be a symplectomorphism. Below we shall also see that the Mess map is a symplectomorphism explicitly (at the origin of both spaces).

3 Maximal surfaces in AdS_3 and universal Teichmüller space

As a preparation for our consideration of the non-compact setting, we start by reviewing the relation between the universal Teichmüller space and maximal surfaces in AdS_3 described in [18]. We shall also present some known facts relating maximal surfaces in AdS_3 , harmonic maps and minimal Lagrangian maps between the unit disc, see [19].

We start by reviewing some details about the universal Teichmüller space. We follow most conventions of [21, 22], and the reader is advised to consult these references for more details. In general terms, the universal Teichmüller space is the space of complex structure

on the unit disc Δ and can be realized as the space of certain equivalence classes of bounded Beltrami coefficients on Δ . More concretely, the definition goes as follows. Let

$$L^\infty(\Delta)_1 = \left\{ \mu : \Delta \rightarrow \mathbb{C}; |\mu|_\infty = \sup_{\Delta} |\mu(w)| < 1 \right\},$$

be the unit ball in the space of bounded Beltrami coefficients in the unit disc Δ . Given $\mu, \nu \in L^\infty(\Delta)_1$ one solves the Beltrami equation

$$\tilde{\mu} = \partial_{\bar{w}} z / \partial_w z$$

in $\hat{\mathbb{C}}$ with coefficients extended by reflection

$$\tilde{\mu}(w) = \begin{cases} \overline{\mu(1/\bar{w})} w^2 / \bar{w}^2, & w \in \hat{\mathbb{C}} \setminus \Delta, \\ \mu(w), & w \in \Delta \end{cases}$$

similarly for ν . The equivalence relation between μ, ν is then given if the corresponding solutions, normalized to fix $-1, -i$ and 1 , agree on \mathbb{S}^1

$$z_\mu|_{\mathbb{S}^1} = z_\nu|_{\mathbb{S}^1}. \quad (9)$$

This is the so-called A-model of the universal Teichmüller space. The complex structure in $\mathcal{T}(1)$ is not easily described in this model, but become much more explicit in the so-called B-model, which works with solutions of the Beltrami equation where the Beltrami coefficient is extended to the outside of the unit disc as $\mu = 0$. Some more facts about the two models and their relation are described in Section 7.

Since the boundary values of quasiconformal diffeomorphisms of the disc are quasisymmetric homeomorphisms of the unit circle, we have an identification between the universal Teichmüller space $\mathcal{T}(1)$ and the space $\text{QS}(\mathbb{S}^1)/\text{Möb}(\mathbb{S}^1)$ of Möbius normalized quasisymmetric homeomorphisms of \mathbb{S}^1 . Thus, for purposes of this section it is sufficient to think about $\mathcal{T}(1)$ as the space of (normalized to fix $-1, -i, 1$) quasisymmetric homeomorphisms of the circle. We note that this space is a symplectic manifold. The symplectic structure is described in details in the Appendix.

Before we explain a relation between $\mathcal{T}(1)$ and the maximal surfaces in AdS_3 , we would like to describe how the fixed topology Teichmüller spaces $\mathcal{T}(\Sigma)$ can be realized as submanifolds on $\mathcal{T}(1)$. This is achieved by restricting the Beltrami coefficients introduced above to have fixed periodicity properties with respect to some “base point” Fuchsian group. Thus, let Γ be a fixed Fuchsian group such that the quotient Δ/Γ is a Riemann surface of the required topology. We then define a space of Beltrami coefficients for Γ

$$L^\infty(\Delta, \Gamma)_1 = \left\{ \mu \in L^\infty(\Delta)_1 : \mu \circ \gamma \frac{\bar{\gamma}'}{\gamma'} = \mu, \forall \gamma \in \Gamma \right\}. \quad (10)$$

We then define the Teichmüller space

$$\mathcal{T}(\Gamma) = L^\infty(\Delta, \Gamma)_1 / \sim, \quad (11)$$

where the equivalence relation is the same as the one introduced above, see (9). It can be shown that this is just the space of all fixed topology Fuchsian groups, which arise as

$$\Gamma_\mu = z_\mu \circ \Gamma \circ z_\mu^{-1}. \quad (12)$$

Thus, we have given a description of the Teichmüller space $\mathcal{T}(\Sigma)$ as a ball of radius one in the space of Beltrami coefficients. The center of this ball is the surface Δ/Γ . Note that the group Γ can be chosen to be rather arbitrary here. One possible choice is that Δ/Γ is a compact surface of given genus. However, a choice where Δ/Γ is an infinite area surface with hyperbolic ends is also possible. This latter choice is the one relevant for the description of multi-black holes.

We would also like to note, without going into much details, that it is possible to generalize the discussion to the include case of non-orientable spatial topology. Fuchsian groups should then be replaced by the so called non-euclidean crystallographic (NEC) groups, discrete groups of isometries of Δ including orientation reversing elements. The invariance property of Beltrami coefficients under these additional elements should then be $\mu \circ \gamma(\bar{\gamma}'/\gamma') = \bar{\mu}$. Then every Klein surface Σ has an orientable complex double cover Σ^c and the Teichmüller space of Σ embeds as an open submanifold of $\mathcal{T}(\Sigma^c)$. We refer the reader to [23] for an exposition on Kleinian surfaces. Thus, if desired, the universal Teichmüller space construction of this paper also includes the “geon” spacetimes studied in e.g. [20].

We now describe a relation between points in $\mathcal{T}(1)$ and maximal surfaces in AdS_3 , established in [18]. The key idea here is that, given a quasisymmetric map $\mathbb{S}^1 \rightarrow \mathbb{S}^1$, its graph in $\mathbb{S}^1 \times \mathbb{S}^1$ can be viewed as a spacelike curve on the conformal boundary ∂AdS_3 . Indeed, as we recalled in the previous section, ∂AdS_3 is ruled by two families of left and right null geodesics, and is therefore essentially the product $\mathbb{S}^1 \times \mathbb{S}^1$. Now, given a spacelike curve on ∂AdS_3 , it is shown there is a unique maximal surface in AdS_3 intersecting the conformal boundary along this curve. The existence part here is quite general and makes only very weak assumptions about the curve at infinity. It is in the proof of the uniqueness that the boundary curves that are graphs of quasisymmetric maps become relevant. Thus, [18] introduces the notion of a width $w(\Gamma)$ of the convex hull $C(\Gamma)$ in AdS_3 of the boundary curve Γ . This width is the supremum of the (time) distance between the upper and lower boundaries of the convex hull. It is then shown that for any boundary curve the width is at most $\pi/2$. The width is strictly less than $\pi/2$ if and only if the boundary curve is the graph of a quasisymmetric map $\mathbb{S}^1 \rightarrow \mathbb{S}^1$. It is then shown that the corresponding maximal surface has sectional curvature bounded above by a negative constant. Finally, based on convexity properties, it is shown that the maximal surface with uniformly negative sectional curvature is unique among complete maximal graphs with given boundary curve at infinity and bounded sectional curvature.

The existence and uniqueness of maximal surfaces in AdS_3 corresponding to points in $\mathcal{T}(1)$, i.e. quasisymmetric maps, can then be seen to be equivalent to the existence and uniqueness of quasiconformal minimal Lagrangian extensions of quasisymmetric homeomorphisms of \mathbb{S}^1 to the interior of the disc. This last point is essentially the same construction as occurs in the compact setting, see the previous section, where a maximal surface in AdS_3 gives rise to a minimal Lagrangian diffeomorphism between the Riemann surfaces (Σ, I_+) and (Σ, I_-) . The same construction extends to the non-compact setting, as was used in [19], and allowed [18] to prove the existence and uniqueness of minimal Lagrangian extensions of quasisymmetric maps.

4 The generalized Mess parameterization of the AdS_3 phase space

In this section we describe the universal phase space of globally hyperbolic AdS spacetimes as parameterized by two copies of the universal Teichmüller space. This is essentially a generalization of the inverse of the Mess map, as described towards the end of section 2. The moduli space of fixed spatial topology AdS_3 manifolds is then obtainable from the universal phase space by restricting the Beltrami coefficients to be invariant under appropriate topology discrete subgroups of $\text{SL}(2, \mathbb{R})$.

Thus, let us take two points in $\mathcal{T}(1)$, which, we remind the reader, can be thought of as two (normalized to fix $-1, 1, -i$) quasisymmetric homeomorphisms of the circle. Let us denote these homeomorphisms by $f_{\pm} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. As will become clear below, it will be useful to think about f_{\pm} as the boundary values of two quasi-conformal maps z_{\pm} from an “origin” disc. Note that we require both z_+ and z_- to map the unit disc into itself; we are thus working in the context of the A-model of the universal Teichmüller space. We call the complex coordinate in the origin disc w , see Figure 2. The maps z_{\pm} are then interpreted as deformations of the origin disc and define complex coordinates, which we also call z_{\pm} , on the Δ_{\pm} discs. Each of the discs Δ_{\pm} has its standard hyperbolic metric

$$I_{\pm} = \frac{4|dz_{\pm}|^2}{(1 - |z_{\pm}|^2)^2}. \quad (13)$$

Note that I_{\pm} , although hyperbolic, are inequivalent to the w -coordinate Poincaré metric. In fact, when pulled back to the origin disc the metrics I_{\pm} become quite nontrivial

$$I_{\pm} = \frac{4|\partial_w z_{\pm}|^2}{(1 - |z_{\pm}(w)|^2)^2} |dw + \mu_{\pm} d\bar{w}|^2,$$

where $\mu_{\pm} = \partial_{\bar{w}} z_{\pm} / \partial_w z_{\pm}$ are the corresponding Beltrami coefficients.

A word is in order about which quasiconformal extensions z_{\pm} of quasisymmetric boundary maps f_{\pm} are considered. Indeed, there are many quasi-conformal maps z_{\pm} in the same equivalence class in the sense of universal Teichmüller theory, i.e. having the same restrictions f_{\pm} to the circle. We shall see below that for our purposes the composition $z_- \circ z_+^{-1}$ will have to satisfy a certain property which makes it unique (given the boundary values). Apart from this restriction, the extensions z_{\pm} are arbitrary, and for our construction of the phase space it will not matter which specific extension is chosen.

Let us now consider $f = f_- \circ f_+^{-1}$ obtained by composing the homeomorphisms f_{\pm} . This is also a quasisymmetric homeomorphism of \mathbb{S}^1 , and according to [18] there is a unique maximal surface (with uniformly negative sectional curvature) in AdS_3 whose boundary curve is the graph of f . The two generalized Gauss maps from this maximal surface to the hyperbolic disc are harmonic, see [19], and their composition is a minimal Lagrangian diffeomorphism. In fact, this is how [18] proves the existence and uniqueness of the minimal Lagrangian extension of a quasisymmetric map. Let $F : \Delta_+ \rightarrow \Delta_-$ denote this minimal Lagrangian diffeomorphism. We now fix the arbitrariness in z_{\pm} (to some extent) by requiring their composition $z_- \circ z_+^{-1}$ to agree with the minimal Lagrangian diffeomorphism F , whose existence and uniqueness is guaranteed by the theorem of [18]:

$$F = z_- \circ z_+^{-1}. \quad (14)$$

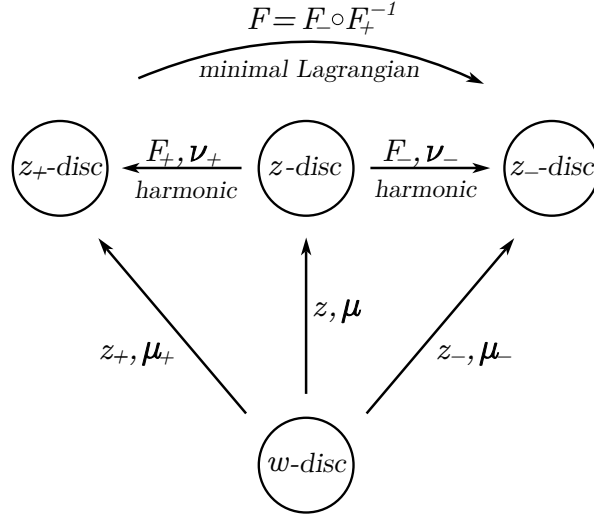


Figure 2: The diagram of maps

Note that this is a condition on the maps z_{\pm} rather than on F , the latter being completely fixed by the boundary quasisymmetric maps f_{\pm} .

As in the compact case, the knowledge of F and the hyperbolic metric on e.g. the disc Δ_+ are enough to reconstruct both the first and second fundamental forms on the maximal surface. Formula (8) gives their expressions evaluated at the z_+ -disc. This description, however, completely hides one direction in the phase space. To obtain the complete description with the dependence on both (f_+, f_-) explicit, we take the pull-back to the origin disc. The nature of the two deformation directions in our phase space then becomes clear. The “difference” $f = f_- \circ f_+^{-1}$ defines a maximal surface in AdS_3 , with its induced first and second fundamental forms and the corresponding isothermal coordinate. Then the particular way f decomposes into f_{\pm} determines a conformal structure of the maximal surface (or a choice of the complex coordinate on the maximal surface possibly different from the isothermal coordinate z there).

To write down the first and second fundamental forms on the maximal surface in terms of the complex coordinate w on the base disc, we need to describe the operator $b : T\Delta_+ \rightarrow T\Delta_+$ associated with the minimal Lagrangian diffeomorphism F . Then, the inverse Mess map (8) directly gives the first and second fundamental forms of the corresponding maximal surface. It is possible to write b explicitly as

$$b = |\partial F| (\partial_{z_+} dz_+ + \lambda \partial_{z_+} d\bar{z}_+ + \bar{\lambda} \partial_{\bar{z}_+} dz_+ + \partial_{\bar{z}_+} d\bar{z}_+)$$

where $\lambda = \partial_{\bar{z}_+} F / \partial_{z_+} F$ is the Beltrami coefficient of F and

$$|\partial F| = \frac{1 - |z_+|^2}{1 - |F|^2} |\partial_{z_+} F|$$

is the holomorphic energy density of F . When F is minimal Lagrangian this operator satisfies the conditions (1-4) in the definition of the inverse of the Mess map and we can construct the fundamental forms of the maximal surface as in the compact case

$$I = \frac{|\partial F|(|\partial F| + 1)}{(1 - |z_+|^2)^2} (2|dz_+|^2 + \bar{\lambda} dz_+^2 + \lambda d\bar{z}_+^2).$$

One then computes the almost-complex structure of I and the operator $(E + b)^{-1}(E - b)$ directly

$$J = i|\partial F| (\partial_{z_+} dz_+ + \lambda \partial_{z_+} d\bar{z}_+ - \bar{\lambda} \partial_{\bar{z}_+} dz_+ - \partial_{\bar{z}_+} d\bar{z}_+)$$

and

$$(E + b)^{-1}(E - b) = -\frac{|\partial F|}{|\partial F| + 1} (\lambda \partial_{z_+} d\bar{z}_+ + \bar{\lambda} \partial_{\bar{z}_+} dz_+).$$

Therefore the second fundamental form II is given by

$$II = -i \frac{|\partial F|}{(1 - |z_+|^2)^2} (\bar{\lambda} dz_+^2 - \lambda d\bar{z}_+^2).$$

These determines the first and second fundamental forms on the maximal surface, written in terms of the coordinate z_+ on the disc Δ_+ .

We now pull-back (I, II) to the origin w -disc using the map z_+ . The transformation of the Beltrami coefficient λ and the holomorphic energy density $|\partial F|$ are easily obtained computing derivatives of $z_- = F \circ z_+$

$$\lambda \circ z_+ \frac{\partial_{\bar{w}} \bar{z}_+}{\partial_w z_+} = \frac{\mu_- - \mu_+}{1 - \mu_- \bar{\mu}_+}, \quad |\partial F| \circ z_+ |\partial_w z_+| = |\partial z_-| \frac{|1 - \bar{\mu}_+ \mu_-|}{1 - |\mu_+|^2}.$$

Using the area preserving condition for F , we get the following expressions

$$I = \frac{1}{(1 - |w|^2)^2} \left(\frac{1}{2} \frac{|\partial z_+|^2}{1 - |\mu_-|^2} + \frac{1}{2} \frac{|\partial z_-|^2}{1 - |\mu_+|^2} + \frac{|\partial z_+| |\partial z_-|}{|1 - \mu_- \bar{\mu}_+|} \right) \left(2(1 - |\mu_-|^2 |\mu_+|^2) |dw|^2 \right. \\ \left. + (\bar{\mu}_+(1 - |\mu_-|^2) + \bar{\mu}_-(1 - |\mu_+|^2)) dw^2 + (\mu_+(1 - |\mu_-|^2) + \mu_-(1 - |\mu_+|^2)) d\bar{w}^2 \right) \quad (15)$$

and

$$II = \frac{i}{(1 - |w|^2)^2} \frac{|\partial z_+| |\partial z_-|}{|1 - \mu_- \bar{\mu}_+|} \left(2(\mu_+ \bar{\mu}_- - \bar{\mu}_+ \mu_-) |dw|^2 \right. \\ \left. + (\bar{\mu}_+(1 + |\mu_-|^2) - \bar{\mu}_-(1 + |\mu_+|^2)) dw^2 - (\mu_+(1 + |\mu_-|^2) - \mu_-(1 + |\mu_+|^2)) d\bar{w}^2 \right). \quad (16)$$

Here

$$|\partial z_{\pm}| = \frac{1 - |w|^2}{1 - |z_{\pm}|^2} |\partial_w z_{\pm}|, \quad (17)$$

are the holomorphic energy densities of z_{\pm} . The pair (I, II) satisfies the Gauss-Codazzi equations, and can be used to construct the spacetime metric via (5).

We close our discussion of the parameterization of AdS_3 spacetimes by two copies of $\mathcal{T}(1)$ by a more detailed discussion of the ambiguity that entered into the above construction. Indeed, recall that the maps z_{\pm} are arbitrary quasiconformal extensions of the boundary quasisymmetric maps f_{\pm} , with the condition that $z_- \circ z_+^{-1}$ is the minimal Lagrangian diffeomorphism extending $f_- \circ f_+^{-1}$. One can now see that nothing depends on the remaining extension ambiguity. Indeed, choosing two different extensions for f_{\pm} , say \tilde{z}_{\pm} , which are in the same universal Teichmüller class as z_{\pm} and still satisfy $\tilde{z}_- \circ \tilde{z}_+^{-1} = F$, we obtain two pairs (I, II) and (\tilde{I}, \tilde{II}) , as well as the corresponding spacetime metrics. It is however clear that the corresponding spacetimes can be mapped on into another by a diffeomorphism that is asymptotically trivial (since the diffeomorphism $\tilde{z}_+ \circ z_+^{-1}$ is asymptotically trivial). Therefore these spacetimes should be considered equivalent and nothing in the above construction depends on which particular extension of f_{\pm} are chosen, provided the minimal Lagrangian condition on $z_- \circ z_+^{-1}$ holds.

5 The generalized Mess map

The aim of this section is to describe an analog of the cotangent bundle parameterization of our phase space. To this end, we first verify that there exist a special decomposition of minimal Lagrangian diffeomorphisms in terms of harmonic maps with opposite Hopf differentials. The corresponding coordinate in the source disc is then shown to be the isothermal coordinate on the maximal surface. The Hopf differentials are just (i times) plus or minus the quadratic differential parameterizing the cotangent direction. These facts are not new and are contained in [19]. We give them here for completeness. We then describe what can be called a generalized Mess map from $T^*\mathcal{T}(1)$ to two copies of $\mathcal{T}(1)$. This arises precisely in the same way as in the compact setting, see (6), the only non-trivial point being the existence and uniqueness of a solution to the Gauss equation (4), which follows from results of [24]. In Section 9 we give a physicists proof that this map is a symplectomorphism. The arising description of a one-to-one symplectomorphism between $T^*\mathcal{T}(1)$ and $\mathcal{T}(1) \times \mathcal{T}(1)$ is a new result, as far as we are aware.

Any minimal Lagrangian diffeomorphism F of the unit disc can be (uniquely) decomposed as a composition $F = F_- \circ F_+^{-1}$ of two harmonic maps F_\pm whose Hopf differentials add up to zero, see e.g. Lemma 2.1 of [19]. As we shall now see, it is this decomposition that leads to the particularly simple expressions for the data on the maximal surface. Let us refer to the coordinate on the source disc of F_\pm by z , see Figure 1. The Hopf differentials are then given by

$$\text{Hopf}(F_\pm) = \frac{4\partial_z F_\pm \partial_z \bar{F}_\pm}{(1 - |F_\pm|^2)^2} dz^2. \quad (18)$$

Using the associated Beltrami differentials

$$\nu_\pm = \partial_z F_\pm / \partial_z F_\pm,$$

we can write these Hopf differentials as

$$\text{Hopf}(F_\pm) = \frac{4|\partial F_\pm|^2 \nu_\pm}{(1 - |z|^2)^2} dz^2, \quad (19)$$

where

$$|\partial F_\pm| = \frac{1 - |z|^2}{1 - |F_\pm|^2} |\partial_z F_\pm|$$

are the corresponding holomorphic energy densities. The Hopf differentials are required to add up to zero, which gives

$$\nu_+ = -\frac{|\partial F_-|^2}{|\partial F_+|^2} \nu_-.$$

The area preserving condition for F reduces to

$$\frac{|\partial F_-|^4}{|\partial F_+|^4} |\nu_-|^2 + \frac{|\partial F_-|^2}{|\partial F_+|^2} (1 - |\nu_-|^2) - 1 = 0.$$

This implies

$$\frac{|\partial F_-|^2}{|\partial F_+|^2} = 1$$

and, in particular, $\nu_+ = -\nu_- = \nu$. The fundamental forms (15), (16) of the maximal surface thus become

$$I = \frac{4|\partial F_+|^2}{(1 - |z|^2)^2} |dz|^2, \quad II = \frac{i}{2} \left(\text{Hopf}(F_+) - \overline{\text{Hopf}(F_+)} \right). \quad (20)$$

Note that the functions F_\pm are not holomorphic or anti-holomorphic in z , and so the metric I is not hyperbolic, despite its seeming resemblance to $4|dF_+|^2/(1 - |F_+|^2)^2$. The formulas (20) are also contained in Proposition 3.1 of [19].

It is clear that what we have obtained is just the cotangent bundle description with the conformal factor and the holomorphic quadratic differential given by

$$e^{2\varphi} = \frac{4|\partial F_+|^2}{(1 - |z|^2)^2}, \quad qdz^2 = i\text{Hopf}(F_+).$$

We note that the Gauss-Codazzi equations for φ and qdz^2 follow directly from the harmonicity of F_+

$$\partial_z \partial_{\bar{z}} F_+ + \frac{2\overline{F_+}}{1 - |F_+|^2} \partial_z F_+ \partial_{\bar{z}} F_+ = 0.$$

We have thus seen that the generalized Gauss maps (6), from the data on the maximal surface to the hyperbolic discs Δ_\pm , are harmonic. This allowed for a simple description (20) of the map $\mathcal{T}(1) \times \mathcal{T}(1) \rightarrow T^*\mathcal{T}(1)$. This fact also allows for a description of the map in the opposite direction. Thus, given a point $(\mu, qdz^2) \in T^*\mathcal{T}(1)$ one can solve for two harmonic maps F_\pm with prescribed Hopf differentials

$$i\text{Hopf}(F_\pm) = \pm q.$$

The existence and uniqueness of harmonic maps with prescribed Hopf differentials was given in [24] by proving that there exists a unique solution of the Gauss equation (4) such that the right- and left-hand-sides are non-negative and such that $e^{2\varphi}|dz|^2$ is a complete metric. This then allows to construct the harmonic maps explicitly via the Mess map (6). For more details of this construction the reader can consult [24]. We note that, although the treatment in this reference is carried for CMC surfaces in the Minkowski space $\mathbb{R}^{1,2}$, it needs very little adaptation to the present situation.

Finally, writing $\nu_\pm = \pm\nu$ for the corresponding Beltrami coefficients, it is now just a matter of using the group structure of $\mathcal{T}(1)$ to get the Beltrami coefficients of the maps from the origin disc:

$$\mu_\pm = \frac{\mu \pm \nu \circ z(\partial_{\bar{w}} \bar{z} / \partial_w z)}{1 \pm \bar{\mu} \nu \circ z(\partial_{\bar{w}} \bar{z} / \partial_w z)}.$$

This gives an explicit description of the generalized Mess map $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$. With maps in both directions being one-to-one, we have demonstrated that the generalized Mess map is a bijection.

6 Relation to the Fefferman-Graham description: the infinitesimal case

In this section we relate the description (5) of spacetimes as evolving data (15), (16) on the maximal surface to the Fefferman-Graham type description of asymptotically AdS_3 space-

times presented in the introduction. Here we treat the infinitesimal case only, and relate the objects appearing in our parameterization to functions a_{\pm} in (2).

The infinitesimal version of the metric (5) with (15), (16) is given by

$$ds^2 = ds_{\text{AdS}_3}^2 + \frac{2 \cos^2 \tau}{(1 - |w|^2)^2} [(\delta \bar{\mu}_+ + \delta \bar{\mu}_-) dw^2 + (\delta \mu_+ + \delta \mu_-) d\bar{w}^2] \\ + \frac{2i \sin \tau \cos \tau}{(1 - |w|^2)^2} [(\delta \bar{\mu}_+ - \delta \bar{\mu}_-) dw^2 - (\delta \mu_+ - \delta \mu_-) d\bar{w}^2] \quad (21)$$

where

$$ds_{\text{AdS}_3}^2 = -d\tau^2 + \frac{4 \cos^2 \tau |dw|^2}{(1 - |w|^2)^2}. \quad (22)$$

In the expression for the metric above μ_{\pm} are the infinitesimal Beltrami coefficients for two quasiconformal maps z_{\pm} from the base w -disc into itself. To compare with the Feffermann-Graham holographic description it is necessary to rewrite the metric in terms of the boundary values of the corresponding quasiconformal maps. Thus, let z_{\pm} be the quasi-conformal maps for $\delta\mu_{\pm}$. These maps being infinitesimal we can write

$$z_{\pm} = w + \delta z_{\pm} + \dots, \quad (23)$$

where the first variations δz_{\pm} are solutions of the corresponding first variations of Beltrami equation

$$\partial_{\bar{w}} \delta z_{\pm} = \delta \mu_{\pm}. \quad (24)$$

Using this relation we can rewrite the metric (21) in terms of the infinitesimal maps δz_{\pm} .

Another fact that we need for our computation is an identity from [25] that follows from the fact that the infinitesimal quasiconformal maps $w + \delta z_{\pm}$ are area preserving. For each of these two maps, the identity reads:

$$\text{Re} \frac{\partial}{\partial w} \frac{\delta z}{(1 - |w|^2)^2} = 0, \quad (25)$$

or, expanding and multiplying by $(1 - |w|^2)$ we have

$$2 \frac{\bar{w} \delta z + w \delta \bar{z}}{(1 - |w|^2)} + \partial_w \delta z + \partial_{\bar{w}} \delta \bar{z} = 0. \quad (26)$$

To compare the metric (21) arising in the maximal surface description with that in the Fefferman-Graham setting, see below, we could have just applied the same coordinate transformation that puts the AdS metrics (22) into the Fefferman-Graham form to the infinitesimal part of the metric in (21). The arising metric could be expected to be of the Fefferman-Graham type, and then one could read off the quantities a, b in terms of the Beltrami coefficients μ_{\pm} . However, this direct way of performing the computation seems to be too difficult, and so we proceed in a different way.

Let us consider a general vector field

$$\xi = \xi^{\tau} \partial_{\tau} + \xi^w \partial_w + \xi^{\bar{w}} \partial_{\bar{w}}$$

written in the coordinates relevant for the maximal surface description. We would like to find a vector field such that Lie derivative of the AdS metric (22) with respect to it is the infinitesimal metric in (21). We will then equate (asymptotically) the vector field that we found with the Brown-Henneaux vector field, see below. This will allow us to related the Beltrami coefficients μ_{\pm} to the parameters appearing in the Brown-Henneaux vector field. From this we will obtain a relation between the parameters in the metric (21) and those in the metric (3).

The Lie derivative of $ds_{\text{AdS}_3}^2$ in the direction of ξ is

$$\begin{aligned} -2\partial_{\tau}\xi^{\tau}d\tau^2 + \left(\frac{4\cos^2\tau\partial_{\tau}\xi^{\bar{w}}}{(1-|w|^2)^2} - 2\partial_w\xi^{\tau}\right)d\tau dw + \left(\frac{4\cos^2\tau\partial_{\tau}\xi^w}{(1-|w|^2)^2} - 2\partial_{\bar{w}}\xi^{\tau}\right)d\tau d\bar{w} \\ + \frac{4\cos^2\tau\partial_w\xi^{\bar{w}}dw^2}{(1-|w|^2)^2} + \frac{4\cos^2\tau\partial_{\bar{w}}\xi^wd\bar{w}^2}{(1-|w|^2)^2} \\ + \frac{4\cos^2\tau}{(1-|w|^2)^2} \left(2\frac{\bar{w}\xi^w + w\xi^{\bar{w}}}{(1-|w|^2)} + \partial_w\xi^w + \partial_{\bar{w}}\xi^{\bar{w}} - 2\tan\tau\xi^{\tau}\right)|dw|^2 \end{aligned}$$

We wish to equate this tensor with the infinitesimal part of the metric in (21), which is the difference between the general metric and the AdS₃ metric

$$\begin{aligned} ds^2 - ds_{\text{AdS}_3}^2 = \frac{2\cos^2\tau}{(1-|w|^2)^2} \left[((1+i\tan\tau)\delta\bar{\mu}_+ + (1-i\tan\tau)\delta\bar{\mu}_-)dw^2 \right. \\ \left. + ((1-i\tan\tau)\delta\mu_+ + (1+i\tan\tau)\delta\mu_-)d\bar{w}^2 \right] \end{aligned} \quad (27)$$

This leads to the followig set of equations

$$\begin{aligned} \partial_{\tau}\xi^{\tau} = 0, \quad \frac{4\cos^2\tau\partial_{\tau}\xi^{\bar{w}}}{(1-|w|^2)^2} - 2\partial_w\xi^{\tau} = 0, \\ 2\frac{\bar{w}\xi^w + w\xi^{\bar{w}}}{(1-|w|^2)} + \partial_w\xi^w + \partial_{\bar{w}}\xi^{\bar{w}} - 2\tan\tau\xi^{\tau} = 0, \\ 2\partial_{\bar{w}}\xi^w = (1-i\tan\tau)\delta\mu_+ + (1+i\tan\tau)\delta\mu_-. \end{aligned} \quad (28)$$

In view of (24), the last equation is clearly satisfied by

$$\xi^w = \frac{1}{2}(1-i\tan\tau)\delta z_+ + \frac{1}{2}(1+i\tan\tau)\delta z_- = \frac{1}{2}(\delta z_+ + \delta z_-) + \frac{1}{2i}\tan\tau(\delta z_+ - \delta z_-)$$

Then the third equation gives

$$\begin{aligned} 2\tan\tau\xi^{\tau} = \frac{\bar{w}(\delta z_+ + \delta z_-) + w(\delta\bar{z}_+ + \delta\bar{z}_-)}{(1-|w|^2)} + \frac{1}{2}\partial_w(\delta z_+ + \delta z_-) + \frac{1}{2}\partial_{\bar{w}}(\delta\bar{z}_+ + \delta\bar{z}_-) \\ + \tan\tau \left(\frac{1}{i} \frac{\bar{w}(\delta z_+ - \delta z_-) - w(\delta\bar{z}_+ - \delta\bar{z}_-)}{(1-|w|^2)} + \frac{1}{2i}\partial_w(\delta z_+ - \delta z_-) - \frac{1}{2i}\partial_{\bar{w}}(\delta\bar{z}_+ - \delta\bar{z}_-) \right) \end{aligned}$$

and using the identity (26) for each δz_{\pm} we have

$$\begin{aligned} \xi^{\tau} = \frac{1}{2i} \frac{\bar{w}(\delta z_+ - \delta z_-) - w(\delta\bar{z}_+ - \delta\bar{z}_-)}{(1-|w|^2)} + \frac{1}{4i}\partial_w(\delta z_+ - \delta z_-) - \frac{1}{4i}\partial_{\bar{w}}(\delta\bar{z}_+ - \delta\bar{z}_-) \\ = \frac{1}{i} \frac{\bar{w}(\delta z_+ - \delta z_-)}{(1-|w|^2)} + \frac{1}{2i}\partial_w(\delta z_+ - \delta z_-) \end{aligned}$$

Therefore the generator of the infinitesimal metric has components

$$\begin{aligned}\xi_T^\tau &= \frac{1}{i} \frac{\bar{w}(\delta z_+ - \delta z_-)}{(1 - |w|^2)} + \frac{1}{2i} \partial_w (\delta z_+ - \delta z_-), \\ \xi_T^w &= \frac{1}{2} (\delta z_+ + \delta z_-) + \frac{1}{2i} \tan \tau (\delta z_+ - \delta z_-)\end{aligned}\tag{29}$$

This gives us an expression for what can be seen as a Brown-Henneaux vector field in the maximal surface description.

Let us now compare this to the holographic description that was sketched in the Introduction. We find it convenient to work with a radial coordinate $\chi = \log(1/\rho)$. The metric (3) can then be written as

$$ds^2 = \frac{e^{2\chi}}{4} (-dt^2 + d\theta^2) + d\chi^2 + \frac{1}{2} (adt^2 + 2bdt d\theta + ad\theta^2) + \frac{e^{-2\chi}}{4} (a^2 - b^2) (-dt^2 + d\theta^2). \tag{30}$$

The Brown-Henneaux vector fields [1], generators of the group of asymptotic symmetries, are parametrized by two functions on the boundary

$$\begin{aligned}\xi_{BH}^t &= \frac{1}{2} (\xi_+ + \xi_-) + e^{-2\chi} (\partial_+^2 \xi_+ + \partial_-^2 \xi_-) + O(e^{-4\chi}) \\ \xi_{BH}^\chi &= -\frac{1}{2} (\partial_+ \xi_+ + \partial_- \xi_-) + O(e^{-4\chi})\end{aligned}\tag{31}$$

$$\xi_{BH}^\theta = \frac{1}{2} (\xi_+ - \xi_-) - e^{-2\chi} (\partial_+^2 \xi_+ - \partial_-^2 \xi_-) + O(e^{-4\chi})$$

with $\xi_\pm = \xi_\pm(t \pm \theta)$.

An infinitesimal version of the metric (30) is given by

$$ds^2 = ds_{\text{AdS}_3}^2 + \frac{1}{2} (\delta a dt^2 + 2\delta b dt d\theta + \delta a d\theta^2),$$

where the infinitesimal part is obtained by applying the Lie derivative with respect to the Brown-Henneaux vector field to the AdS_3 metric given by

$$ds_{\text{AdS}_3}^2 = -\cosh^2 \chi dt^2 + d\chi^2 + \sinh^2 \chi d\theta^2,$$

and corresponding to $b = 0, a = -1$ in (30). The arising relation between the perturbations $\delta a, \delta b$ and the functions ξ_\pm parameterizing the vector field (31) is given by:

$$\begin{aligned}\delta a &= -\frac{1}{2} (\partial_+ \xi_+ + \partial_+^3 \xi_+) - \frac{1}{2} (\partial_- \xi_- + \partial_-^3 \xi_-), \\ \delta b &= -\frac{1}{2} (\partial_+ \xi_+ + \partial_+^3 \xi_+) + \frac{1}{2} (\partial_- \xi_- + \partial_-^3 \xi_-).\end{aligned}\tag{32}$$

We now relate the two descriptions using the fact that they represent the same spacetime in different coordinates. This can be done in the neighbourhood of the boundary curve of the maximal surface where both the Fefferman-Graham and the coordinates of (5) are defined.

Let us compute the components of the Brown-Henneaux vector fields (31) in the coordinates used in (22). The coordinate transformation relating the AdS_3 metric in the form (22) and its equidistant coordinates description is

$$\tan t = \frac{1 - |w|^2}{1 + |w|^2} \tan \tau, \quad \sinh \chi = \frac{2|w|}{1 - |w|^2} \cos \tau, \quad \theta = \arg w.$$

We can therefore get for the Brown-Henneaux vector field (31) in equidistant coordinates:

$$\begin{aligned} \xi_{BH}^\tau &= \frac{1 + |w|^2}{1 - |w|^2} \xi_{BH}^t + \frac{2|w| \sin \tau}{[(1 - |w|^2)^2 + 4|w|^2 \cos^2 \tau]^{1/2}} \xi_{BH}^\chi \\ \xi_{BH}^w &= w \tan \tau \xi_{BH}^t - \frac{1}{2} \frac{w}{|w|} \frac{(1 - |w|^4) \sec \tau}{[(1 - |w|^2)^2 + 4|w|^2 \cos^2 \tau]^{1/2}} \xi_{BH}^\chi + iw \xi_{BH}^\theta. \end{aligned} \quad (33)$$

We only need the asymptotic behavior of these components, evaluated on the $\tau = 0$ surface. Thus, we get, for the leading terms

$$\xi_{BH}^\tau = \frac{\xi_+ + \xi_-}{1 - |w|^2} + \dots, \quad \xi_{BH}^w = \frac{iw}{2} (\xi_+ - \xi_-) + \dots, \quad (34)$$

where the dots stand for the subleading components.

On the other hand, the maximal surface description vector fields (evaluated on the $\tau = 0$ surface) are, asymptotically

$$\xi_T^\tau = \frac{1}{i} \bar{w} \frac{\delta z_+ - \delta z_-}{1 - |w|^2} + \dots, \quad \xi_T^w = \frac{1}{2} (\delta z_+ + \delta z_-). \quad (35)$$

We now equate

$$\xi_{BH}^\tau = \xi_T^\tau, \quad \xi_{BH}^w = \xi_T^w, \quad (36)$$

and then immediately read off the sought relations:

$$\delta z_\pm \Big|_{|w|=1} = \pm iw \xi_\pm. \quad (37)$$

7 The Feffermann-Graham stress tensor and the Bers embedding

In this section we develop an interpretation of the relation (37) between the maximal surface and the holographic descriptions. We remain at the infinitesimal level, leaving the question of relating the general evolving data (15), (16) metric and the Fefferman-Graham metric (30) to future studies.

Thus, we have seen how the two functions ξ_\pm parameterizing the Brown-Henneaux vector fields are related to the boundary values of the quasi-conformal maps δz_\pm in the maximal surface description. However, a more interesting question is that of a relation between the holographic stress-energy tensor components – functions a, b in (30) – and the quasi-conformal maps. In this section we shall see that this relation is that between the holomorphic quadratic differentials arising via the so-called Bers embedding of $\mathcal{T}(1)$ and the quasiconformal maps

parameterizing $\mathcal{T}(1)$. In other words, here we shall see that the stress-energy tensor components of the holographic description are nothing else but the components of the quadratic differentials that arise via the Bers embedding of $\mathcal{T}(1) \times \mathcal{T}(1)$.

Let us start by reminding the reader some facts about the Bers embedding. We are necessarily brief here, and for more details the reader can consult a very accessible exposition in [26]. The Bers embedding arises via the so-called B-model of the universal Teichmüller space. Let us recall that so far we have been solving the Beltrami equation $\partial_{\bar{w}}z = \mu\partial_wz$ for a quasiconformal map z_μ starting from a (bounded) Beltrami coefficient μ in the unit disc, and then extending μ symmetrically to the outside of the disc. This gives rise to a quasiconformal map z_μ that maps the inside of the unit disc into itself (and thus the unit circle into itself and the outside into itself), and reduces to a quasisymmetric map on the unit circle. This gave rise to the A-model of the universal Teichmüller space. This model hides the complex structure of $\mathcal{T}(1)$, as the maps z_μ resulting from this construction depend only real-analytically on μ .

The B-model arises by taking the same bounded Beltrami coefficients inside the unit disc, and then extending them to be zero outside of Δ . The corresponding quasi-conformal map z^μ is conformal outside of the disc, it is in fact biholomorphic into its image, and depends on μ complex-analytically. The universal Teichmüller space in the B-model is defined to be the space of equivalence classes of Beltrami coefficients (or quasi-conformal maps), with two maps called equivalent if they agree outside of the unit disc (as well as on the unit circle). Taking the Schwarzian derivative of the map z^μ conformal outside of Δ one obtains a holomorphic quadratic differential outside of the unit disc. This is the Bers embedding of $\mathcal{T}(1)$ to the space of holomorphic quadratic differentials outside of Δ .

At the infinitesimal level, there exists an explicit relation between the two models exhibited in [26]. Thus, the tangent vectors to $\mathcal{T}(1)$ in the A-model can be described as functions $u : \mathbb{S}^1 \rightarrow \mathbb{R}$ on the circle that are related to the so-called Zygmund class functions on the real line, see [26] and references therein for more details. What is important for us here is that the functions u are defined from the boundary values of the corresponding infinitesimal quasi-conformal maps δz via:

$$u(e^{i\theta}) = \frac{\delta z(e^{i\theta})}{ie^{i\theta}}. \quad (38)$$

The functions arising as tangent vectors to $\mathcal{T}(1)$ are those for which

$$F(x) = \frac{1}{2}(x^2 + 1)u\left(\frac{x - i}{x + i}\right) \quad (39)$$

are of the Zygmund class, see [26]. The functions u can be expanded into a Fourier series:

$$u(e^{i\theta}) = \sum_{k=-\infty}^{\infty} u_k e^{ik\theta}. \quad (40)$$

The fact that $u(e^{i\theta})$ is real implies $u_{-k} = \overline{u_k}$, where the overline denotes the complex conjugation.

The tangent space to the B-model universal Teichmüller space is described as follows. In this model the solution z^μ of the Beltrami equation is of the form $z^\mu(w) = w + \delta z(w)$, with

the function δz now being holomorphic outside of the unit disc. It admits an expansion

$$w + \delta z(w) = w \left(1 + \frac{c_2}{w^2} + \frac{c_3}{w^3} + \dots \right) \quad \text{in } |w| > 1, \quad (41)$$

where a Möbius transformation is used to remove the $1/w$ term in the brackets and to set the first term to unity. The holomorphic quadratic differential that is obtained as the (infinitesimal) Schwarzian derivative $\partial_w^3 \delta z$ of (41) admits an expansion

$$h(w) = \frac{1}{w^4} \left(h_0 + \frac{h_1}{w} + \frac{h_2}{w^2} + \dots \right) \quad \text{in } |w| > 1. \quad (42)$$

The coefficients h_k are related to those in (41) via:

$$h_{k-2} = c_k(k - k^3), \quad k \geq 2. \quad (43)$$

Finally, the relation between the A- and B-model Fourier coefficients u_k and c_k is given by, see [26]

$$c_k = i\overline{u_k}, \quad (44)$$

where, as before, the overline denotes the complex conjugation.

For the later purposes, we now note that in all the discussions of the B-model above we could have replaced the outside of the unit disc with the inside. In fact, this is the choice in some of the references, see e.g. [21], [22]. In this case one works with bounded Beltrami differentials outside of the disc, solves the Beltrami equation continuing $\mu = 0$ inside Δ , and gets a holomorphic function inside the disc, whose Schwarzian derivative produces a holomorphic quadratic differential in Δ . We could have as well worked with this modes for the universal Teichmüller space. In fact, as we shall see below, it will be natural to work with holomorphic functions outside for one copy of $\mathcal{T}(1)$ and with holomorphic functions inside Δ for the other copy from $\mathcal{T}(1) \times \mathcal{T}(1)$. The analogs of (41) and (42) in this realization of $\mathcal{T}(1)$ are

$$w + \delta \hat{z}(w) = w(1 + \hat{c}_2 w^2 + \hat{c}_3 w^3 + \dots), \quad |w| < 1, \quad (45)$$

and

$$\hat{h}(w) = \hat{h}_0 + \hat{h}_1 w + \hat{h}_2 w^2 + \dots \quad |w| < 1, \quad (46)$$

where we have denoted the quantities arising in this realization of the B-model by letters with an extra hat. We also note that there is an extra minus as compared to (43) in the relation between the coefficients in this realization: $\hat{h}_{k-2} = -\hat{c}_k(k - k^3)$. Finally, we note that we can always map a holomorphic function inside the disc to an anti-holomorphic function outside by $w \rightarrow 1/\bar{w}$. By applying this to the quadratic differential (46) we get a new anti-holomorphic quadratic differential outside of the disc

$$h(\bar{w}) = \overline{\hat{h}\left(\frac{1}{w}\right)} \frac{1}{w^4} = \frac{1}{\bar{w}^4} \left(\bar{\hat{h}}_0 + \frac{\bar{\hat{h}}_1}{\bar{w}} + \frac{\bar{\hat{h}}_2}{\bar{w}^2} + \dots \right) \quad \text{in } |w| > 1, \quad (47)$$

which is the same expression as in (42), but with the change $w \rightarrow \bar{w}$. We shall use this realization in terms of anti-holomorphic functions outside of Δ for the second copy of $\mathcal{T}(1)$, and the “usual” realization in terms of holomorphic functions outside Δ for the first copy of the universal Teichmüller.

Another relation we need is that between the c -coefficients in the realization of the B-model in terms of Beltrami coefficients outside of the disc (we have denoted these coefficients by \hat{c}_k above), and the Fourier coefficients u_k of the Zygmund functions u of the A-model. Formula (44) gives such relation for one realization of the B-model, and we need to derive a similar relation for the other realization. The derivation is a straightforward adaptation of the Proof I in [26]. One finds

$$\hat{c}_k = -\frac{1}{\pi} \int_{\Delta^*} \frac{\mu}{z^{k+2}} dx dy, \quad (48)$$

where the integral is carried out over the complement Δ^* of the unit disc Δ . In the integrand here μ is the Beltrami coefficient of the B-model in the realization where it is non-zero outside of the unit disc. In relating it to the A-model Fourier coefficients we use the fact that the A-model Beltrami coefficient outside of the disc is obtained by the reflection from that inside the disc. Thus, the integral in (48) can be rewritten as one over the unit disc, where the integrand will depend on the reflected Beltrami

$$\mu\left(\frac{1}{\bar{w}}\right) = \overline{\mu(w)} \frac{w^2}{\bar{w}^2}. \quad (49)$$

Substituting this to the integral, and taking into account the change of integration measure $dx dy \rightarrow -dx dy/w^2 \bar{w}^2$ we get

$$\hat{c}_k = \frac{1}{\pi} \int_{\Delta} \overline{\mu(w)} \bar{w}^{k-2} dx dy, \quad (50)$$

which is just the complex conjugation of the coefficients of the other, interior of the unit disc realization of the B-model

$$\hat{c}_k = \overline{c_k}. \quad (51)$$

We can now continue to think about both points in $\mathcal{T}(1) \times \mathcal{T}(1)$ as being parameterized by Beltrami coefficients inside Δ . This gives two A-model quasi-conformal maps in Δ whose boundary values produce two Zygmund functions u . Expanding these into Fourier modes we get two sets of coefficients u_k , which we shall later denote by u_k^\pm . Now our convention is that the B-model for the first copy of $\mathcal{T}(1)$ is obtained by setting to zero the Beltrami outside of Δ , thus producing holomorphic functions in Δ^* , and the quadratic differentials as in (42). The B-model for the second copy of $\mathcal{T}(1)$ will be realized by setting to zero the Beltrami coefficient inside the disc, and thus producing a holomorphic differential inside Δ , which in turn can be interpreted as an anti-holomorphic quadratic differential in Δ^* , as in (47). Collecting all the relations above we can write the following relations between the u - and h -coefficients of the two copies of $\mathcal{T}(1)$:

$$h_{k-2}^\pm = \pm i \overline{u_k^\pm} (k - k^3), \quad (52)$$

where we have now differentiate between the two copies of $\mathcal{T}(1)$ by assigning one with a label plus and another minus. The notation now is that in (47) $\hat{h}_{k-2}^- = h_{k-2}^-$.

Now, to prepare for the relation between the Bers embedding quadratic differentials and the holomorphic stress-energy tensor that we will derive, let us rewrite the stress tensor

$$T = adt^2 + 2bdtd\theta + ad\theta^2 \quad (53)$$

of the Fefferman-Graham metric in a suggestive way. To this end, we will analytically continue the t coordinate to the imaginary values. Thus, let us continue all the functions appearing in T via

$$t = \frac{1}{2i} \log |w|^2, \quad \theta = \frac{1}{2i} \log \frac{w}{\bar{w}}, \quad (54)$$

so that the new (imaginary) time coordinate runs between $-i\infty$ and $i\infty$, while w runs over the complex plane. The unit circle $|w| = 1$ corresponds to the circle $t = 0$. With this choice we have

$$t + \theta = \frac{1}{i} \log w, \quad t - \theta = \frac{1}{i} \log \bar{w}, \quad (55)$$

so that functions of $t \pm \theta$ become holomorphic (anti-holomorphic) functions on the complex plane. In particular, the functions a_{\pm} whose sum and difference give a, b would seem to become a holomorphic and anti-holomorphic function on the complex plane. However, there is no (bounded) holomorphic function on the whole complex plane apart from a constant. Thus, we need to be very careful when designing the analytic continuation. So, let us expand a_{\pm} into Fourier modes. When restricted to $t = 0$ these are periodic functions of θ , and so the Fourier expansion is possible. We have

$$a_{\pm}(t \pm \theta) = \sum_{k=-\infty}^{\infty} a_k^{\pm} e^{ik(t \pm \theta)}. \quad (56)$$

As before, we have $\overline{a_k^{\pm}} = a_{-k}^{\pm}$ so that these are real functions. It is clear that we cannot continue a_{\pm} as holomorphic or anti-holomorphic functions into the whole complex plane, but what is possible is to take what can be called the chiral part of a_{\pm} , this containing only, say, the negative frequency modes, and continue only this part. Thus, let us introduce

$$\tilde{a}_{\pm}(t \pm \theta) = \sum_{k=-\infty}^{-2} a_k^{\pm} e^{ik(t \pm \theta)} \quad (57)$$

for the chiral parts. Here we have used the fact that $|k| \geq 2$ in these expansions, which will become manifest below. We now continue the chiral parts via (54) to the exterior of the unit disc to get:

$$\tilde{a}_+(w) = \frac{a_{-2}^+}{w^2} + \frac{a_{-3}^+}{w^3} + \dots, \quad \tilde{a}_-(\bar{w}) = \frac{a_{-2}^-}{\bar{w}^2} + \frac{a_{-3}^-}{\bar{w}^3} + \dots, \quad (58)$$

which are a holomorphic and anti-holomorphic functions outside Δ respectively. We can now analytically continue the chiral part \tilde{T} of the stress tensor (46), which is the tensor T with functions a_{\pm} replaced by their chiral parts. A simple computation gives:

$$\tilde{T} = -\tilde{a}_+(w) \frac{dw^2}{w^2} - \tilde{a}_-(\bar{w}) \frac{d\bar{w}^2}{\bar{w}^2}. \quad (59)$$

We shall now relate the chiral parts \tilde{a}_\pm of the stress-energy tensor to the functions arising via the Bers embedding of $\mathcal{T}(1) \times \mathcal{T}(1)$. To this end, let us first obtain a relation between the Fourier coefficients a_k^\pm and those of the Fourier expansions of the functions ξ_\pm appearing in the Brown-Henneaux vector fields. We have:

$$-2a_\pm = \partial_\pm \xi_\pm + \partial_\pm^3 \xi_\pm, \quad (60)$$

and so if we expand

$$\xi_\pm(t \pm \theta) = \sum_{k=-\infty}^{\infty} \xi_k^\pm e^{ik(t \pm \theta)}, \quad (61)$$

with $\xi_{-k}^\pm = \overline{\xi_k^\pm}$, we get the following relation between the Fourier coefficients.

$$-2a_k^\pm = i(k - k^3)\xi_k^\pm. \quad (62)$$

We now come back to the problem of relating the maximal surface and holographic descriptions. We have seen that the relation between the Brown-Henneaux functions ξ_\pm and the boundary values of the quasiconformal maps δz_\pm is given by (37). In view of (38) this can be rewritten as

$$\xi_\pm(\pm\theta) = \pm u_\pm(e^{i\theta}), \quad (63)$$

where on the left-hand-side the functions ξ_\pm are restricted to the circle $t = 0$. This implies the following relation between the the Fourier coefficients of the ξ -functions and the A-model functions u_\pm

$$\xi_k^\pm = \pm u_k^\pm, \quad (64)$$

with $u_{-k}^\pm = \overline{u_k^\pm}$. We can therefore write $-2a_k^\pm = \pm i u_k^\pm (k - k^3)$ or

$$2a_{-k}^\pm = \pm i \overline{u_k^\pm} (k - k^3). \quad (65)$$

Comparing this with (52) we see that

$$h_{k-2}^\pm = 2a_{-k}^\pm, \quad k \geq 2. \quad (66)$$

This gives us the desired relation between the coefficients in the expansions (58) of the chiral parts of the stress-energy tensor and those of the Beltrami quadratic differentials (42), (47). This relation implies that

$$\frac{2\tilde{a}_+(w)}{w^2} = h^+(w), \quad \frac{2\tilde{a}_-(\bar{w})}{\bar{w}^2} = h^-(\bar{w}), \quad (67)$$

and that, finally, the analytic continuation of the chiral part \tilde{T} of the stress-energy tensor is equal to (minus half) the sum of two quadratic differentials arising via the Bers embedding:

$$\tilde{T} = -\frac{1}{2}h^+(w)dw^2 - \frac{1}{2}h^-(\bar{w})d\bar{w}^2. \quad (68)$$

This is our final result for the (infinitesimal) relation between the maximal surface and the holographic descriptions.

8 Charges

In this section we use relation (68) obtained above to derive an expression for the asymptotic charges for a general spacetime in our maximal surface parameterization. Similar to the previous section, we shall do so at the infinitesimal level. However, the answer that we obtain admits an obvious generalization to the finite case. We shall see that the charges are given simply by the (real parts of) the periods of the Bers embedding quadratic differentials.

We start by computing the charges of an asymptotically AdS spacetime in the holographic Fefferman-Graham description. Here one starts with the Einstein-Hilbert action complemented with the York-Gibbons-Hawking boundary term and a volume renormalization counterterm

$$S = -\frac{1}{2} \int_M d^3x \sqrt{-g} (R - 2\Lambda) - \int_{\partial M} d^2x \sqrt{-\gamma} (\Theta + 1) \quad (69)$$

The quasi-local stress-energy tensor is then obtained from the variation of the action with respect to the boundary metric

$$T_{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{\mu\nu}}. \quad (70)$$

One gets

$$T_{\mu\nu} = \Theta_{\mu\nu} - \Theta^\rho{}_\rho \gamma_{\mu\nu} - \gamma_{\mu\nu} \quad (71)$$

where γ is the boundary metric and Θ the boundary extrinsic curvature.

Let us now consider a spacetime that is asymptotically described by a Fefferman-Graham type metric (30). Let M_{χ_0} denote the portion of the spacetime manifold where $\chi < \chi_0$. The metric induced on $\chi = \chi_0$ surface is

$$\gamma = \frac{1}{4} e^{2\chi_0} (-dt^2 + d\theta^2) + \frac{1}{2} (adt^2 + 2bdt d\theta + ad\theta^2) + \frac{1}{4} e^{-2\chi_0} (a^2 - b^2) (-dt^2 + d\theta^2), \quad (72)$$

with

$$\sqrt{-\gamma} = \frac{1}{4} (e^{2\chi} - e^{-2\chi} (a^2 - b^2)) \quad (73)$$

The extrinsic curvature of the $\chi = \chi_0$ surface is given by

$$\Theta_{\mu\nu} = -\mathcal{L}_n \gamma_{\mu\nu}|_{\chi_0} = -\nabla_\mu n_\nu|_{\chi_0} = -\frac{1}{2} \partial_\chi g_{\mu\nu}|_{\chi_0} \quad (74)$$

where $n = -\partial_\chi$ is the unit normal vector field to the boundary ∂M_{χ_0} . Therefore

$$\Theta = -\frac{1}{4} [e^{2\chi_0} - e^{-2\chi_0} (a^2 - b^2)] (-dt^2 + d\theta^2), \quad (75)$$

with

$$\text{Tr}(\Theta) = -2 \frac{e^{2\chi_0} + e^{-2\chi_0} (a^2 - b^2)}{e^{2\chi_0} - e^{-2\chi_0} (a^2 - b^2)}. \quad (76)$$

Putting all the ingredients together we get

$$T = adt^2 + 2bdt d\theta + ad\theta^2. \quad (77)$$

We now take a spacelike slice $\Sigma = \{t = 0\}$ and compute its unit normal timelike vector field

$$u = \frac{2}{e^{2\chi} - e^{-2\chi}(a^2 - b^2)} \left[\sqrt{e^{2\chi} + 2a + e^{-2\chi}(a^2 - b^2)} \partial_t - \frac{2b}{\sqrt{e^{2\chi} + 2a + e^{-2\chi}(a^2 - b^2)}} \partial_\theta \right]$$

Now consider $\Sigma_{\chi_0} = M_{\chi_0} \cap \Sigma$. The induced metric on its boundary $\partial\Sigma_{\chi_0}$ is

$$\sigma = \frac{1}{4} [e^{2\chi_0} + 2a + e^{-2\chi_0}(a^2 - b^2)] d\theta^2 \quad (78)$$

For each asymptotic Killing vector field ξ we have a conserved charge

$$Q_\xi = \lim_{\chi_0 \rightarrow \infty} \frac{1}{2\pi} \int_{\partial\Sigma_{\chi_0}} d\theta \sqrt{|\sigma|} u^\mu \xi^\nu T_{\mu\nu}. \quad (79)$$

Mass is the conserved charge associated with time translation ($\xi = \partial_t$):

$$M = \lim_{\chi_0 \rightarrow \infty} \frac{1}{2\pi} \int_{\partial\Sigma_{\chi_0}} d\theta \sqrt{|\sigma|} (u^0 T_{00} + u^2 T_{20}) = \frac{1}{2\pi} \int_{\partial\Sigma} d\theta a(\theta). \quad (80)$$

Angular momentum is the conserved charge associated with rotation ($\xi = -\partial_\theta$)

$$J = \lim_{\chi_0 \rightarrow \infty} \frac{1}{2\pi} \int_{\partial\Sigma_{\chi_0}} d\theta \sqrt{|\sigma|} (u^0 T_{02} + u^2 T_{22}) = \frac{1}{2\pi} \int_{\partial\Sigma} d\theta b(\theta). \quad (81)$$

Taking into account the relations

$$a(t, \theta) = a_+(t + \theta) + a_-(t - \theta), b(t, \theta) = a_+(t + \theta) - a_-(t - \theta)$$

to the chiral functions a_\pm we can rewrite the above formulas for the charges compactly as

$$\frac{1}{2}(M \pm J) = \frac{1}{2\pi} \int_{\partial\Sigma} a_\pm. \quad (82)$$

Note that, even prior to any relation to the maximal surface description, these can be expressed as the real parts of the periods of the holomorphic quadratic differentials arising via the analytic continuation of the chiral parts of a_\pm , see the previous section.

We now relate this to the maximal surface description. From (67) we know that the chiral parts of the functions a_\pm are basically the (analytic continuations of the) quadratic differentials h_\pm arising from the Bers embedding. The full functions a_\pm on the circle can be obtained by taking their chiral parts and adding the complex conjugate. Thus, we have $2\text{Re}(\tilde{a}^\pm)|_{|w|=1} = a^\pm(\theta)$ and therefore

$$\frac{1}{2}(M + J) = \frac{1}{2\pi} \text{Re} \oint_{|w|=1} w^2 h^+, \quad \frac{1}{2}(M - J) = \frac{1}{2\pi} \text{Re} \oint_{|w|=1} \bar{w}^2 h^-. \quad (83)$$

which are just the (real parts of the) periods of the Bers embedding quadratic differentials h^\pm . The formulas in terms of the Bers embedding quadratic differentials are of course only valid at the infinitesimal level, where we have a relation between the functions a_\pm of the holographic description and the data on the maximal surface. However, as we noted above, the same formulas are valid even in the finite case if one understands that h_\pm are the (multiples of the) analytic continuations of the chiral parts of a_\pm , see (67). It is then natural to conjecture that the analytic continuations of the chiral parts of a_\pm continue to be related to the Bers embedding quadratic differentials in the same way as they do in the infinitesimal case, and that (83) gives a general formula for the charges in terms of the maximal surface data. We leave an attempt at demonstration this finite case relation to future work. We also note that in this infinitesimal case the charges (83) are actually zero, for there is no $1/w^2, 1/\bar{w}^2$ terms in the expansion of the infinitesimal quadratic differentials h^\pm , see (42), (47). So, the first order variation of the charges, computed at the origin corresponding to the AdS_3 is zero. It is clear however that considering non-trivial spatial topologies, obtained as the quotients of AdS_3 by some discrete groups of isometries, will render non-trivial periods for the (anti-)holomorphic quadratic differentials and therefore non-trivial charges in each asymptotic region.

9 The phase space symplectic structure

We can now describe the gravitational symplectic structure on our phase space. We first compute the symplectic structure in the cotangent bundle description, and then translate it into the generalized Mess description $\mathcal{T}(1) \times \mathcal{T}(1)$ using the Mess map. We shall see that the pull-back of the canonical cotangent bundle symplectic structure on $T^*\mathcal{T}(1)$ to $\mathcal{T}(1) \times \mathcal{T}(1)$ coincides with the difference of Weil-Petersson symplectic structures in each copy of $\mathcal{T}(1)$. Thus, the generalized Mess map is symplectic. We do computations by comparing the symplectic structures at the origin of both spaces. The result at an arbitrary point should then follow using the group structure of the universal Teichmüller space $\mathcal{T}(1)$, but we shall not attempt to demonstrate this in the present paper. Instead, next section shows the Chern-Simons $SL(2, R)$ connections are respectively parametrized by μ_\pm , which then implies the gravitational symplectic structure should coincide with the difference of Weil-Petersson symplectic structures in each sector.

From the Hamiltonian formulation of general relativity, one knows that the pre-symplectic 1-form is given by

$$\Theta = \frac{1}{2} \int_{\Delta} (\mathbb{I}, \delta I)_I da_I = \frac{1}{2} \int_{\Delta} \text{tr}(I^{-1} \mathbb{I} I^{-1} \delta I) da_I.$$

When working in the maximal surface with its first and second fundamental forms given by

$$I = e^{2\varphi} |dz|^2, \quad \mathbb{I} = \frac{1}{2} (q dz^2 + \bar{q} d\bar{z}^2),$$

we have for the first variation of I :

$$\delta I = e^{2\varphi} (\delta \bar{\mu} dw^2 + \delta \mu d\bar{w}^2 + (2\delta\varphi + \partial_w \delta z + \partial_{\bar{w}} \delta \bar{z}) |dw|^2).$$

Here μ is the Beltrami differential describing variations of the conformal structure of the maximal surface. The pre-symplectic 1-form is therefore

$$\Theta = \int_{\Delta} d^2w (q\delta\mu + \bar{q}\delta\bar{\mu}).$$

We see that the holomorphic quadratic differential determining the second fundamental form is canonically conjugated to the variable μ parameterizing the conformal structure of the maximal surface. We note that it is the same computation that is valid in the context of compact spatial sections AdS_3 manifolds and in our context of asymptotically AdS_3 spacetimes. Taking the variation of the pre-symplectic 1-form we get

$$\Omega = \int_{\Delta} d^2w (\delta q \wedge \delta\mu + \delta\bar{q} \wedge \delta\bar{\mu}),$$

which shows that the symplectic structure induced by the Einstein-Hilbert functional is just the canonical cotangent bundle symplectic structure on $T^*\mathcal{T}(1)$.

Now, using the Mess map we can write the variations of μ and q at the origin (corresponding to AdS_3) in terms of those of μ_{\pm}

$$\delta\mu = \frac{1}{2}(\delta\mu_+ + \delta\mu_-), \quad \delta q = \frac{4i}{(1-|w|^2)^2}\delta\bar{\nu} = \frac{2i}{(1-|w|^2)^2}(\delta\bar{\mu}_+ - \delta\bar{\mu}_-).$$

The gravitational symplectic form, evaluated at the origin of our phase space, therefore becomes

$$\Omega = \frac{1}{2i} \int_{\Delta} \frac{4d^2w}{(1-|w|^2)^2} (\delta\mu_+ \wedge \delta\bar{\mu}_+ - \delta\mu_- \wedge \delta\bar{\mu}_-), \quad (84)$$

which is just a copy of the Weil-Petersson symplectic form in each $\mathcal{T}(1)$. This shows the generalized Mess map $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$ (at the origin of both spaces) is symplectic. It should be possible to extend this to an arbitrary point by using the group structure of the universal Teichmüller space, see the appendix, but we shall not attempt this here.

The symplectic structure (84), as written, seems to depend on the Beltrami differentials in the disc, and thus on the maps z_{\pm} . Thus, it seems that there is some arbitrariness in how the tangent vectors (at the origin) are represented as infinitesimal Beltrami coefficients. However, as is well-known and is reviewed in the Appendix, there is a canonical representative of the tangent vectors as the so-called harmonic Beltrami coefficients. This can be evaluated explicitly in terms of the Fourier coefficients of the quasimetric maps that parameterize the tangent vectors in the A-model. Each copy of the Weil-Petersson symplectic structure can then be written explicitly in terms of the Fourier coefficients, see the Appendix.

10 Chern-Simons connections

Finally, in this Section, we present a relation between our $\mathcal{T}(1) \times \mathcal{T}(1)$ parametrization of the phase space and Chern-Simons formulation of 2+1 general relativity [9]. In the first order formalism the variables one works with are a frame field e and a spin connection ω . For

negative cosmological constant, these may be combined into a $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ connection $A = (A^+, A^-)$ on $M = \mathbb{R} \times \Delta$

$$A_\mu^\pm = (\omega_\mu^a \pm e_\mu^a) T_a,$$

where T_a are the generators of $\text{SL}(2, \mathbb{R})$

$$T_0 = \frac{i}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_1 = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad T_2 = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

so that we have

$$\text{tr}(T_a T_b) = \frac{1}{2} \eta_{ab}, \quad [T_a, T_b] = \epsilon_{ab}{}^c T_c.$$

Written in terms of A^+ and A^- , the Einstein-Hilbert action becomes the difference of two Chern-Simons action

$$S_{EH}[A^+, A^-] = S_{CS}[A^+] - S_{CS}[A^-].$$

In this sense we say that 2+1 GR is equivalent to Chern-Simons theory. Note, however, that the phase space of Chern-Simons theory, which is the space of all solutions of the equations of motion, is much bigger than that of GR. Basically, some connections define non invertible frames e so that singular metrics are also included. The gauge group of CS theory also includes some transformations (large gauge transformations) that cannot be considered as gauge from the point of GR. In spite of this, the CS point of view on AdS_3 gravity is very convenient, because it gives the simplest way to understand how the Mess-type description by two copies of the Teichmüller space can be possible.

The CS formulation thus shows there exists a pair of flat $\text{SL}(2, \mathbb{R})$ connections associated with any AdS metric. To relate our phase space construction to CS theory we compute the flat $\text{SL}(2, \mathbb{R})$ connections associated with the AdS metric parametrized by (f_+, f_-) . Again, it is convenient to start working at the maximal surface. The 3-metric can then be written

$$ds^2 = -d\tau^2 + \cos^2 \tau e^{2\varphi} |dz|^2 + \sin \tau \cos \tau (q dz^2 + \bar{q} d\bar{z}^2) + \sin^2 \tau e^{-2\varphi} |q|^2 |dz|^2$$

and it is a simple computation to find the associated flat $\text{SL}(2, \mathbb{R})$ connections

$$(a_\pm)_z = \frac{1}{2} \begin{bmatrix} \partial_z \varphi & \mp e^\varphi \\ i e^{-\varphi} q & -\partial_z \varphi \end{bmatrix}, \quad (a_\pm)_{\bar{z}} = \frac{1}{2} \begin{bmatrix} -\partial_{\bar{z}} \varphi & -i e^{-\varphi} \bar{q} \\ \mp e^\varphi & \partial_{\bar{z}} \varphi \end{bmatrix}.$$

Here we have eliminated the τ dependence using a gauge transformation. Recalling that the Liouville field and the holomorphic quadratic differential can be written, in terms of F_\pm , as

$$e^{2\varphi} = \frac{4|\partial F_+|^2}{(1 - |z|^2)^2}, \quad q dz^2 = i \text{Hopf}(F_+),$$

we need just another gauge transformation $a_\pm \rightarrow g^{-1} a_\pm g + g^{-1} dg$, with

$$g = \begin{bmatrix} (\partial_z F_\pm / |\partial_z F_\pm|)^{-1/2} & 0 \\ 0 & (\partial_z F_\pm / |\partial_z F_\pm|)^{1/2} \end{bmatrix},$$

to see the connections decouple. A pull-back to the origin disc then gives us

$$(a_{\pm})_w = \frac{1}{(1 - |z_{\pm}|^2)} \begin{bmatrix} \frac{1}{2}(\bar{z}_{\pm}\partial_w z_{\pm} - z_{\pm}\partial_w \bar{z}_{\pm}) & \mp \partial_w z_{\pm} \\ \mp \partial_w \bar{z}_{\pm} & -\frac{1}{2}(\bar{z}_{\pm}\partial_w z_{\pm} - z_{\pm}\partial_w \bar{z}_{\pm}) \end{bmatrix},$$

$$(a_{\pm})_{\bar{w}} = \frac{1}{(1 - |z_{\pm}|^2)} \begin{bmatrix} -\frac{1}{2}(\bar{z}_{\pm}\partial_w z_{\pm} - z_{\pm}\partial_w \bar{z}_{\pm}) & \mp \partial_w \bar{z}_{\pm} \\ \mp \partial_w z_{\pm} & \frac{1}{2}(\bar{z}_{\pm}\partial_w z_{\pm} - z_{\pm}\partial_w \bar{z}_{\pm}) \end{bmatrix},$$

and we see that each copy of $\mathcal{T}(1)$ parametrizes one of the CS sectors, as could have been expected.

11 Discussion

In this paper we described an explicit parameterization of a large class of AdS_3 manifolds by two copies of the universal Teichmüller space $\mathcal{T}(1)$. Our construction proceeds by first determining the first and second fundamental forms on the maximal surface that corresponds to a given point in $\mathcal{T}(1) \times \mathcal{T}(1)$, and then evolving this initial data using (5) to get the spacetime metric. We note that only half of the data in $\mathcal{T}(1) \times \mathcal{T}(1)$ is needed to get the maximal surface with its first and second fundamental forms. The other half of the phase space coordinates determines a complex structure on the maximal surface, which may or not coincide with the complex structure of the isothermal complex coordinate on this surface. Geometrically, half of the data determine the curve along which the maximal surface intersects the boundary at infinity, while the other half determines how the maximal surface itself is foliated by $|w| = \text{const}$ curves, see Fig. 1. We have also seen that an equally good description of the same class of spacetimes is provided by $T^*\mathcal{T}(1)$, and that the generalized Mess map between the two descriptions is a symplectomorphism.

We have then studied the relation between the maximal surface description of AdS_3 spacetimes given in this paper and the more standard holographic description by Fefferman-Graham metrics (30). We have only been able to give an infinitesimal relation between two such metrics that are close to the standard metric on AdS_3 . However, this allowed us to give an expression (83) for the charges (mass and angular momentum) of a spacetime in terms of data on the maximal surface. This expression admits an immediate generalization to the finite case, where the charges would simply be given by the real parts of the periods of the holomorphic quadratic differentials arising from the Bers embedding of $\mathcal{T}(1) \times \mathcal{T}(1)$. It would be very interesting to see that this is indeed the case for a general metric from our family. We leave this to future work. We have also shown that our description in terms of $\mathcal{T}(1) \times \mathcal{T}(1)$ is natural in terms of the Chern-Simons description of AdS_3 gravity, in that the two Chern-Simons connections corresponding to our AdS_3 metrics decouple with each being parameterized by a single copy of $\mathcal{T}(1)$.

The natural question that arises is what our constructions can add to the debate as to the microscopic origin of the entropy of 2+1 dimensional black holes. Here we can only give some speculations on this issue. As we have already mentioned in the Introduction, it seems sensible to approach the problem of quantum gravity in 2+1 dimensions as the problem of quantization of the moduli space of 2+1 dimensional constant curvature manifolds. In the context of negative cosmological constant all fixed spatial topology moduli spaces are

realized as submanifolds of the universal moduli space described in the present work. The universal space therefore includes all possible multi black holes, together with the Brown-Henneaux excitations in each asymptotic region of such a BH. Our phase space also includes all compact spatial slice spacetimes (in this case one should simply take the initial data to be invariant under a Fuchsian group of a compact surface), but this spacetimes are unlikely to be relevant to the problem of BH entropy. One can then reformulate the question of computing the BH entropy as that of computing the partition function over all possible multi black hole spacetimes with fixed mass and angular momentum of one of the asymptotic regions. The entropy could then be extracted from this “canonical” partition function by the standard thermodynamic formulas. Our (infinitesimal case) expression (83) for the charges is then the first step in this direction.

It would be very interesting if it was possible to reformulate the partition function computation as that in the context of some conformal field theory. In this respect we note that the Gauss-Codazzi equations that arise on the maximal surface in AdS_3 are integrable, and are those of the so-called sl_2 affine Toda system. It thus could be that the conformal field theory associated to the sl_2 affine Toda is the CFT relevant for the quantum description of AdS_3 gravity. We note that this CFT would naturally live on the maximal surface, not on the asymptotic boundary. But we have seen that the analytic continuation (to the imaginary time) of the functions on the AdS_3 boundary cylinder has a natural interpretation in terms of data on the maximal surface. Thus, it appears that the Euclidean signature CFT on the spatial slice can be, when analytically continued, be relevant for the AdS/CFT type description of 2+1 dimensional quantum gravity. Whether any of these speculations have a chance to come out true only future works on the subject can tell.

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12 Appendix A: Universal Teichmüller theory

We give a very basic introduction to the theory of universal Teichmüller space trying to keep the work as self-contained as possible. Our presentation follows closely the presentations of [26, 21, 22, 27].

Let $\Delta = \{w \in \hat{\mathbb{C}}; |z| < 1\}$ and $\Delta^* = \{w \in \hat{\mathbb{C}}; |z| > 1\}$ be the unit disc and its exterior in the Riemann sphere $\hat{\mathbb{C}}$ and let

$$L^\infty(\Delta)_1 = \left\{ \mu : \Delta \rightarrow \mathbb{C}; |\mu|_\infty = \sup_{\Delta} |\mu(w)| < 1 \right\},$$

be the unit ball in the space of bounded Beltrami differentials on Δ . The universal Teichmüller space $\mathcal{T}(1)$ is defined as the space of equivalence classes of such bounded Beltrami

coefficients on Δ ,

$$\mathcal{T}(1) = L^\infty(\Delta)_1 / \sim,$$

the equivalence relation being defined as follows.

Model A. Given two bounded Beltrami coefficients $\mu, \nu \in L^\infty(\Delta)_1$ one solves Beltrami equations in \mathbb{C} with coefficients extended to Δ^* by reflection

$$\tilde{\mu}(w) = \begin{cases} \overline{\mu(1/\bar{w})} w^2 / \bar{w}^2, & w \in \Delta^*, \\ \mu(w), & w \in \Delta, \end{cases}$$

similarly for ν . Then μ, ν are taken to be equivalent if the corresponding solutions, normalized to fix $-1, -i$ and 1 , agree in \mathbb{S}^1

$$z_\mu|_{\mathbb{S}^1} = z_\nu|_{\mathbb{S}^1}.$$

Model B. Equivalently, one can define the equivalence relation by solving the Beltrami equations in \mathbb{C} with Beltrami coefficients give by

$$\tilde{\mu}(w) = \begin{cases} 0, & w \in \Delta^*, \\ \mu(w), & w \in \Delta, \end{cases}$$

similarly for ν . Now, μ, ν are considered equivalent if the corresponding solutions, normalized to have a simple pole of residue 1 at ∞ and to satisfy $z(w) - w \rightarrow 0$ for $w \rightarrow \infty$, agree on Δ^*

$$z^\mu|_{\Delta^*} = z^\nu|_{\Delta^*}.$$

One can, therefore, describe universal Teichmüller space either as the space of (normalized) quasiconformal homeomorphisms on \mathbb{S}^1 or the space of (normalized) univalent functions on Δ^* .

Model B allows an embedding, the so called Bers embedding, of $\mathcal{T}(1)$ in the space of holomorphic quadratic differentials on Δ^*

$$A_\infty(\Delta^*) = \{h : \Delta^* \rightarrow \mathbb{C} \text{ holomorphic}; |h(w)(1 - |w|^2)^2|_\infty < \infty\}$$

via Schwarzian derivative of $z^\mu|_{\Delta^*}$

$$\{z^\mu|_{\Delta^*}, w\} = \frac{\partial_w \partial_w \partial_w z^\mu}{\partial_w z^\mu} - \frac{3}{2} \left(\frac{\partial_w \partial_w z^\mu}{\partial_w z^\mu} \right)^2.$$

$L^\infty(\Delta)_1$ also carries a group structure induced by the composition of quasiconformal maps. The group multiplication is defined as $\lambda = \nu * \mu$ iff the following relation is satisfied

$$(\nu \circ z_\mu) = \frac{\lambda - \mu}{1 - \bar{\lambda} \bar{\mu}} \frac{\partial_w z_\mu}{\partial_w \bar{z}_\mu}.$$

More explicitly, λ is the Beltrami coefficient of $z_\lambda = z_\nu \circ z_\mu$ and is given by

$$\lambda = \frac{\mu + \nu \circ z_\mu \frac{\partial_w \bar{z}_\mu}{\partial_w z_\mu}}{1 + \bar{\mu} \nu \circ z_\mu \frac{\partial_w \bar{z}_\mu}{\partial_w z_\mu}}.$$

That such group structure descends to $\mathcal{T}(1)$ is clear from the fact that z_μ, z_ν leave \mathbb{S}^1 invariant.

Tangent Space

The derivative of the projection $L^\infty(\Delta)_1 \rightarrow \mathcal{T}(1)$ at the origin identifies the tangent space $T_{[0]}\mathcal{T}(1)$ with the space of harmonic Beltrami differentials on Δ

$$\Omega^{-1,1}(\Delta) = \left\{ \delta\mu = -\frac{(1-|w|^2)^2}{2} \overline{\hat{h}(w)}; \hat{h} \in A_\infty(\Delta) \right\},$$

where

$$A_\infty(\Delta) = \left\{ \hat{h} : \Delta \rightarrow \mathbb{C} \text{ holomorphic}; |\hat{h}(w)(1-|w|^2)^2|_\infty < \infty \right\}.$$

In model B, this is the induced an isomorphism $T_{[0]}\mathcal{T}(1) \rightarrow A_\infty(\Delta^*)$ by the Bers embedding

$$\delta\mu \mapsto h(w) = -\frac{6}{\pi} \int_\Delta d^2\zeta \frac{\delta\mu(\zeta)}{(\zeta-w)^4}, \quad w \in \Delta^*$$

where $d^2\zeta = \frac{i}{2}d\zeta \wedge d\bar{\zeta}$. Its inverse $A_\infty(\Delta^*) \rightarrow \Omega^{-1,1}(\Delta)$ is given by

$$h \mapsto \delta\mu(w) = -\frac{(1-|w|^2)^2}{2} h(1/\bar{w}) \frac{1}{\bar{w}^4}, \quad w \in \Delta.$$

In model A, given $\delta\mu \in T_{[0]}\mathcal{T}(1)$, we have a one-parameter family of quasiconformal maps z_ϵ satisfying

$$\partial_{\bar{w}} z_\epsilon = \epsilon \delta\mu \partial_w z_\epsilon$$

and fixing $-1, -i, 1$. Since $z_0 = \text{Id}$, we can write, for small ϵ ,

$$z_\epsilon = z + \epsilon \delta z + O(\epsilon^2),$$

where the first variation δz is a solution of

$$\partial_{\bar{w}} \delta z = \delta\mu.$$

Let $h \in A_\infty(\Delta^*)$ be the holomorphic quadratic differential corresponding to $\delta\mu$ by the Bers embedding. Writing

$$h = i \sum_{k \geq 2} \frac{(k-k^3)\overline{u_k}}{w^{k+2}}$$

for the Laurent expansion of h (the choice of coefficients is for later convenience), one can explicitly find δz by integration

$$\begin{aligned} \delta z(w) &= \frac{i}{2} \sum_{k \geq 2} \overline{u_k} \bar{w}^{k-1} [k(k+1) - 2(k^2-1)|w|^2 + k(k-1)|w|^4] + F(w), & w \in \Delta \\ &= \frac{i}{2} \sum_{k \geq 2} u_k \bar{w}^{-k-1} [k(k+1)|w|^4 - 2(k^2-1)|w|^2 + k(k-1)] - w^2 \overline{F(1/\bar{w})}, & w \in \Delta^* \\ &= -w^2 \overline{\delta z(1/\bar{w})} \end{aligned}$$

where F is some holomorphic function on Δ . Writing it as

$$F(w) = \sum_{k \geq 0} v_k w^k$$

and restricting δz to \mathbb{S}^1 we get

$$\begin{aligned} \delta z(e^{i\theta}) &= \sum_{k \geq 2} i \overline{u_k} e^{-(k-1)i\theta} + v_0 + v_1 e^{i\theta} + v_2 e^{2i\theta} + \sum_{k \geq 2} v_{k+1} e^{(k+1)i\theta} \\ &= \sum_{k \geq 2} i u_k e^{(k+1)i\theta} - \overline{v_0} e^{2i\theta} - \overline{v_1} e^{i\theta} - \overline{v_2} - \sum_{k \geq 2} \overline{v_{k+1}} e^{-(k-1)i\theta} \end{aligned}$$

so

$$v_0 = -\overline{v_2}, \quad v_1 = -\overline{v_1}, \quad v_{k+1} = i u_k, \quad k \geq 2$$

and

$$\begin{aligned} \delta z(w) &= \frac{i}{2} \sum_{k \geq 2} \overline{u_k} w^{k-1} [k(k+1) - 2(k^2 - 1)|w|^2 + k(k-1)|w|^4] \\ &\quad + v_0 + v_1 w + v_2 w^2 + i \sum_{k \geq 2} u_k w^{k+1}, \quad w \in \Delta \end{aligned}$$

Note that, because of the normalization condition imposing $\delta z_{\delta\mu}$ to vanish at $-1, -i$ and 1 , the coefficients v_0, v_1, v_2 are completely determined by the $u_k, k \geq 2$. In fact, the Möbius group is realized exactly as

$$\text{Möb}(\mathbb{S}^1) \approx \{u(e^{i\theta}) = \overline{u_1} e^{-i\theta} + u_0 + u_1 e^{i\theta}\}$$

and, therefore, those coefficients are gauge. From now on we will drop the coefficients, understanding that they acquire the necessary values to make δz vanish at $-1, -i, 1$.

One may also think of the family $z_\epsilon|_{\mathbb{S}^1}$ as the one-parameter flow of the vector field $\delta z \partial_z$. Its restriction to \mathbb{S}^1 is

$$\delta z \partial_z|_{\mathbb{S}^1} = u(e^{i\theta}) \partial_\theta$$

with

$$u = \frac{\delta z(e^{i\theta})}{i e^{i\theta}} = \sum_{k \neq -1, 0, 1} u_k e^{ik\theta}.$$

This is an element of the so called Zygmund class on \mathbb{S}^1 , $\Lambda(\mathbb{S}^1)$, defined by

$$\Lambda(\mathbb{S}^1) = \{u : \mathbb{S}^1 \rightarrow \mathbb{R} \text{ continuous such that } A_u \in \Lambda(\mathbb{R})\}$$

where $A_u(x) = \frac{1}{2}(x^2 + 1)u\left(\frac{x-i}{x+i}\right)$ and

$$\begin{aligned} \Lambda(\mathbb{R}) &= \{A : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous such that} \\ &\quad |A(x+t) + A(x-t) - 2A(x)| \leq \kappa|t|, \kappa > 0\}. \end{aligned}$$

Note that the coefficients u_{-1}, u_0, u_1 were dropped due to the normalization condition. Consequently, u belongs to the quotient $\Lambda(\mathbb{S}^1)/\text{Möb}(\mathbb{S}^1)$ and the construction above provides an identification between $T_{[0]} \mathcal{T}(1)$ and the Möbius normalized Zygmund class on \mathbb{S}^1 .

Weil-Petersson Hermitian metric

The almost complex structure at the origin of $\mathcal{T}(1)$ is most clear from the model B point of view in which $J : T_{[0]}\mathcal{T}(1) \rightarrow T_{[0]}\mathcal{T}(1)$ is just

$$Jh = ih.$$

By the isomorphism above described we have the almost complex structure

$$Ju = i \sum_{k \neq -1, 0, 1} \text{sgn}(k) u_k e^{in\theta}$$

on the space of normalized Zygmund class functions.

The Weil-Petersson hermitian metric on Teichmüller space of Riemann surfaces can be easily generalized to a hermitian metric on universal Teichmüller space. Explicitly, given $\delta\mu, \delta\nu \in T_{[0]}\mathcal{T}(1)$ we define

$$\langle \delta\mu, \delta\nu \rangle_{\text{WP}} = \int_{\Delta} \frac{4d^2w}{(1-|w|^2)^2} \delta\mu(w) \overline{\delta\nu(w)}.$$

Bers embedding then gives

$$\langle h, q \rangle_{\text{WP}} = \int_{\Delta} d^2w (1-|w|^2)^2 h(w) \overline{q(w)},$$

for $h, q \in A_{\infty}(\Delta^*)$, and the isomorphism above $\Lambda(\mathbb{S}^1)/\text{Möb}(\mathbb{S}^1) \rightarrow A_{\infty}(\Delta^*)$

$$\langle u, v \rangle_{\text{WP}} = \sum_{k, l \geq 2} k(k^2 - 1)l(l^2 - 1) u_k \overline{v_l} \int_{\Delta} d^2w (1-|w|^2)^2 w^{k-2} \bar{w}^{l-2}.$$

Using

$$\int_{\Delta} d^2w (1-|w|^2)^2 w^{k-2} \bar{w}^{l-2} = \int_{\Delta} dr d\theta (1-r^2)^2 r^{k+l-3} e^{i(k-l)\theta} = \frac{2\pi}{k(k^2 - k)} \delta_{kl}$$

we get the Weil-Petersson symplectic structure in terms of the coefficients of the Zygmund functions

$$\langle u, v \rangle_{\text{WP}} = 2\pi \sum_{k \geq 2} k(k^2 - 1) u_k \overline{v_l}.$$

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